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Abstract

In agricultural economics, several calibration and aggregation approaches have evolved in mathematical programming models. This article combines in a linear programming model features of the Positive Mathematical Programming method with an aggregation approach that is constrained to the production possibility set spanned by a convex combination of observed production activities. The combination is obtained by using a variable separation technique that approximates a non-linear objective function. Therefore, linear programming models can be exactly calibrated to observed production activities. The aggregation of production activities in homogenous production response units assumes that farmers in a region are treated such as they respond in the same way. Both methodologies are embedded in economic reasoning and provide a robust framework to solve large-scale linear programming models in reasonable time.

Key words: calibration, aggregation, linear programming

Introduction

This article presents a methodology to overcome some of the difficulties practitioners have when large-scale models are used to analyse farm policy changes. Mathematical programming models are close to a true model, if the decision making process can be adequately represented such that observed production activities can be reproduced. Some analysts prefer models with a non-linear objective function because responses to policy changes are smooth, unrealistic corner solutions can be prevented and the introduction of flexibility constraints can be avoided. The increasing availability of detailed administrative data, some of them even at farm level, challenges policy analysts to use these adequately. However, the combined complexity of discretionary policies and large numbers of heterogeneous production units frequently prevents that non-linear models can be solved in reasonable time, or sometimes even at all.

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Any modeller has to make choices on techniques and methods to deal with aggregation and calibration. Day (1993) derived a set of conditions that must be met for unbiased aggregation, the problem to represent a group of heterogeneous producers by a single unit. The virtual impossibility of meeting Day's criteria led to several approaches to reduce the aggregation bias, by bottom-up procedures like grouping similar farms (e.g., Buckwell and Hazell, 1972) and by top-down procedures by restricting the choice on crop mix to a convex combination of historical crop mixes (McCarl, 1982).

McCarl (1982) argued that choices of farmers are revealed in observable activities e.g., crop mixes which embed many farm specific constraints and attitudes (e.g., crop rotation, technology, price and policy expectation, etc.). Convex combinations of historical crop mixes reflect optimal choices in an aggregated model that is consistent with farm specific situations. Even if non observed crops are modelled, this method can be employed. In such a case alternative crop mixes are established based on agronomic rules. This approach has been extended to aggregate heterogeneous farm firms (Önal and McCarl, 1989) and was applied in large-scale sector models (McCarl et al., 1993; Adams et al., 1996).

Models should reproduce base-run results to observed production activities and respond realistically to policy and price changes. Different methods have been developed to solve this calibration and aggregation problem. One approach is to impose production economic criteria (marginal revenue equals marginal cost or equality of value marginal products) and to use the shadow price vector of a linear programming model (LP) to derive calibration parameters (Fajardo et al., 1981 and Howitt, 1995). The method suggested by Howitt (1995), Positive Mathematical Programming (PMP), has become widely used to calibrate agricultural production and supply models at various scales i.e., farm, region and sector. However, the resulting non-linear objective function comes to some cost. Solving PMP-models usually takes much longer than purely LP model. Hence, either the models are highly abstract and aggregated, or they are made separable to iteratively approximate some equilibrium state.

The methodology of PMP relies on the assumption that an observed production activity allocation of a farm, or in a region is the consequence of profit maximising behaviour. Observed average cost are used in a three step procedure to derive additional unobservable cost which are compressed into parameters of a non-linear optimization model. In the first phase a perturbed LP model provides activity based duals that are used in phase two to derive calibration coefficients which enter a non-linear objective function of the calibrated model in phase three (Howitt, 1995). Such a calibrated model reproduces exactly an observed crop allocation.

In the proposed alternative approach, phase three is of major interest. We show that the shape of a non-linear objective function can be approximated by using a variable separation technique. An exact calibration using PMP parameters is therefore possible in a LP model.

When the methodology of PMP was published (Howitt, 1995), only the diagonal elements of the additional cost matrix Q were identified. The implicit assumption that off-diagonal elements are zero, means that cross-activity relationships are ignored. So far, the literature provides only small scale examples to derive or estimate off-diagonal elements in the Q matrix. The approximations, employing maximum entropy estimations, are often based on a single observation and are close to zero (Paris and Howitt, 1998; Heckelei and Britz, 1999).

If the values of the elements of the Q matrix are based on a large number of observations more reliable information on the interaction between activities might be revealed. Our article makes an attempt to attain the same goal without estimating the off-diagonal elements by employing the method for top-down aggregation suggested by McCarl (1982).

The article is structured such that the basic idea of combining both methods is illustrated and discussed in the following LP model. Special attention is paid to calibration and aggregation. It finishes with some references to policy evaluation problems for which this method seems to offer a promising approach and an outlook for further methodological developments.

The LP model set-up

Suppose, the objective is to maximize producer surplus (PS) from the production of i crops using v different management practices (e.g., different tillage systems or environmentally friendly management measures such as cover crops) in a region. Revenues are the product of given prices (?) and crop output (?). Production costs (?) are non-linearly increasing in output (Figure 1).

The other letters represent factor uses and other technical characteristics of production (?), historically observed crop mixes (?), and observed resource endowments (?). The choice on crop (i) and management (v) shares is obtained by assigning fractions (?) to a convex set of production increments ($\overline{B}_{i,v,s}$) using the technique of variable separation. Similarly, the crop mix choice (?) is restricted to the set of historical crop mixes (index m).

Output increments (index s) are percentages (?$_s$) of observed output levels (O) ranging, for instance, from 10 to 200 percent. The design of the increments can be such that they are smaller close to the observed level (e.g., O_4 in figure 1) and get sequentially larger the more distant they are.

(1) $$\max_{k,q} PS = \sum_{i,v,s} \left[\left(r_{i,v} * \mathrm{O}_{i,v,s} - \mathrm{X}_{i,v,s} \right) * q_{i,v,s} \right]$$

(2) s.t. $$\sum_{i,v,s} \left(\mathrm{A}_{i,v} * \overline{\mathrm{B}}_{i,v,s} * q_{i,v,s} \right) \leq \sum_{i,v} \left(\mathrm{B}_{i,v} \right)$$

(3) $$\sum_{m} \left(\mathrm{K}_{i,m} * k_m \right) \leq \sum_{v,s} \left(\overline{\mathrm{B}}_{i,v,s} * q_{i,v,s} \right)$$ for all i

(4) $$\sum_{m} \left(k_m \right) = 1$$

(5) $$\sum_{s} \left(q_{i,v,s} \right) = 1$$ for all i and v

(6) $$0 \leq q_{i,v,s}, j_m \leq 1$$

where $\mathrm{X}_{i,v,s} = \int_0^{\overline{\mathrm{B}}_{i,v,s}} \left(a_{i,v} + 2 * b_{i,v} * \overline{\mathrm{B}}_{i,v,s} + 2 * j_{i,v} * \sum_{\tilde{v}} \left(\overline{\mathrm{B}}_{i,\tilde{v},s} \right) \right) d\overline{\mathrm{B}}_{i,v,s}$ are approximated multi-variant production cost increments of quadratic shape that are calculated for each production grid $\overline{\mathrm{B}}_{i,v,s}$. Production girds are computed as $\overline{\mathrm{B}}_{i,v,s} = \mathrm{B}_{i,v} * \Psi_s$. The coefficients of a linearly increasing multi-variant marginal cost curve ($\alpha_{i,v}$, $\beta_{i,v}$, and $\varphi_{i,v}$) are derived in the PMP process (phase 2). The intercept coefficient of the linear multi-variant cost curve is

(7) $$a_{i,v} = 1 - \frac{\left(l_i + l_{i,v} \right)}{VC_{i,v}},$$

the slope coefficient of variant activity levels is

(8) $$b_{i,v} = \frac{l_{i,v}}{VC_{i,v} * \mathrm{B}_{i,v}},$$ and

the slope coefficient of crop activity levels is

(9) $$j_{i,v} = \frac{l_{i,v}}{VC_{i,v} * \sum_{\tilde{v}} \mathrm{B}_{i,\tilde{v}}}.$$

where the λ are modified duals of the perturbed model. For many countries average variable costs (VC) of production activities are usually published by extension services, or derived from farm accounting data, or calculated by farm engineering models.

The model is calibrated to some observed production activity levels ($\mathrm{B}_{i,v}$) using the extended PMP method of variant production technologies developed by Röhm (2001) and Röhm and Dabbert (2003). They argue, that alternative management practices must be considered carefully when environmental effects of policies at regional scales are analyzed. Their method allows an higher substitution between different management technologies (i.e., applying environmentally friendly management measures) than between crops. A reduction of payments for an agri-environmental measure (e.g., cover crop after wheat) will probably lead to a decline of adoption of this management measure. The land under this management is more likely to be allocated to the same crop (e.g., conventionally produced wheat) than to a different crop (e.g., corn). Such an adjustment is facilitated by separate slope coefficients. One depends on the management-variant activity level (ß), and the other on the total crop activity level (f).

Calibration

The PMP method is based on two major conditions: (a) marginal gross margins of each activity are identical in the base-run, and (b) the average PMP gross margins are identical to the average LP gross margins for each activity in the base-run. These conditions guarantee that the objective function values of the perturbed LP model and the calibrated PMP model are almost identical in the base-run.

An assumption must be made concerning the marginal gross margin effect. It must be assigned either to marginal cost, marginal revenue, or fractional to both. In the LP example above, the marginal gross margin effect is assigned to the marginal cost. Consequently, coefficients of linearly increasing multi-variant marginal cost curves are derived.

By definition, the area beneath a linear marginal cost curve is the *variable* cost of production as expressed in $?_{i,v,s}$, or the associated point on quadratic variable cost curve. Total crop output is the product of the observed crop yield per hectare ($?_{i,v}$) with the corresponding production increment ($\mathrm{O}_{i,v,s} = \overline{\mathrm{B}}_{i,v,s} * g_{i,v}$). The convexity and identity condition in equation (5) allows any weighed combination of all production grids ($\overline{\mathrm{B}}_{i,v,s}$). The optimal crop and management shares in hectares are finally computed by $\overline{\mathrm{B}}_{i,v,s} * q^{*}_{i,v,s}$. Similarly, total production output is the sum over all crop outputs ($\mathrm{O}_{i,v,s} * q^{*}_{i,v,s}$), total revenue is the sum of outputs times prices ($r_{i,v} * \mathrm{O}_{i,v,s} * q^{*}_{i,v,s}$), and total variable production costs are the sum of cost increments ($\mathrm{X}_{i,v,s} * q^{*}_{i,v,s}$).

Figure 1 is a graphical illustration of the linear calibration approach using variable separation. As already mentioned, integrating a linear increasing marginal cost curve (e.g., MC = a + 2ß?) will result in a quadratic variable cost curve (VC = c + a? + ß? ^{2}). Cost parameters a and ß are derived in phase 2 of the PMP procedure.

Figure 1

Suppose ?$_4$ is the observed production quantity. An arbitrary set of neighboring production quantities (?$_1$,..., ?$_7$) can be easily calculated using ?. The integration of the MC curve over each production quantity grid (?$_1$,...,?$_7$) will provide a set of corresponding variable production costs (?$_1$,...,?$_7$). Consequently, ?$_1$ is equivalent to the area $\overline{0ah}$, ?$_2$ to area $\overline{0bi}$,..., and ?$_7$ to area $\overline{0gn}$. The production choice (?) is restricted to the set of production grids (?$_1$,...,?$_7$), or alternatively to the set of variable production costs (?$_1$,...,?$_7$) as shown in figure 1. Given the coefficients (a, and ß) of linearly increasing marginal cost curves, the LP model is now calibrated to the observed production activities (i.e., the diagonal elements in the Q matrix of the PMP model).

In this extended model setting of multi-variant production technologies the derived coefficients of the marginal cost curves include an intercept (a). Röhm and Dabbert (2003) did not mention or interpret the possibility of non-zero intercept values. According to our view a positive intercept could be interpreted as a fixed-cost component associated with the production of a particular crop (e.g., non-output related cost for an organic crops certificate). A negative intercept could mean that the production of the crop can not decline beyond the point where MC becomes zero (only the positive part of an increasing MC is considered). This situation could reflect some crop rotational restriction on the farm or in the region. Nevertheless, in the original PMP model developed by Howitt (1995), the MC curve has no intercept. Therefore, to avoid non-zero intercepts, the MC cost curve in figure 1 could be reduced to MC = 2ß?. This would mean that a crop grown under different management technologies is treated now as if it were two separate crops. However, if there is a historically set of alternative technologies mixes available (e.g., mix$_5$ in table 1), one could form a convex combination instead, such as in the crop mix approach (see next).

Aggregation

The aggregation problem is usually based on the assumption that there is a duality between solving an aggregate model that has all the farm models in full detail included, and building an aggregate model without the farm models, that is constrained to the production possibility set spanned by a convex combination of all possible optimal solutions of the farm models (Önal and McCarl, 1991). Because it is practically impossible to construct all the detailed farm models, one can use for instance historical observations on crop mixes instead. Suppose the cumulative production choices of farmers in our model region are revealed by historical crop mixes (?) as listed in table 1. The crop mix choice (?$_m$) is restricted to the set of observed crop mixes (equation 3) (see also table 1). Equation (4) provides that the convexity and identity conditions are fulfilled. The advantage of this approach is that the production possibility set can be expanded beyond historic allocations if non observed crops are important in the analysis (e.g., non-food crops).

Table 1

The set of crop mixes should be preferably large, because by assumption they reveal choices, which contain information on farm specific restrictions and attitudes (e.g., crop rotation, production technology, price and policy expectations, risk attitudes, etc.). One feature of the approach presented in this article is that a change of production cost is accounted for, when the crop mix is changing. The PMP coefficients provide that cost are moving along the curve when a crop acreage changes. Such an adjustment does not take place in the original version of the method presented by McCarl (1982).

Conclusion

Any modeller has to deal with the calibration and aggregation problem in agricultural production and sector models. A model should be calibrated such that it reproduces as closely as possible an observed set of decision maker's actions and the methods to attain this, should be based on economic reasoning. An aggregated model treats a group of producers as if they all responded in the same way as a single representative production unit does. The literature provides linear and non-linear methods for both problems.

In the last few years, PMP has become a commonly used method to calibrate (diagonal elements of the Q matrix) and aggregate (off-diagonal elements in the Q matrix) models at various scales. However, the applicability of this method is limited because a non-linear objective function is resulting in the PMP process, which makes large scale model analyses a time consuming effort or futile at all. In addition, the elements of the off-diagonal Q matrix are usually point estimates that do not necessarily reflect *average* production responses.

Our article contributes to the literature by combining in a LP model, features of the PMP calibration method with an top-down aggregation method that builds convex combinations of historically observed production activities (e.g., crop mixes). Consequently, the diagonal elements in the Q matrix are used for calibrating production activities and the convex combination of crop mixes substitute for estimation of off-diagonal elements. The combination is obtained by using a variable separation technique that approximates a non-linear objective function. In our example, a quadratic objective function is approximated, but the variable separation technique allows any functional form of marginal cost and revenues. This combination of calibration and aggregation methods provides a robust framework suitable for large-scale model analyses.

The combination of these methodologies was initiated by the practical need to evaluate policy experiments in an heterogeneous spatial setting. It was used to analyse the consequences of the 2003 reform of the Common Agricultural Policy (CAP) in Austria, an EU member state where the volume of the rural development programme exceeds the volume of those support payments which have been affected by the reform (Sinabell and Schmid, 2003; Schmid and Sinabell, 2004). The Positive Agricultural Sector Model Austria (PASMA) differentiates production activities with respect to 19 land categories, 36 cash crops, 48 feeding activities and crops, 29 livestock categories, and 34 livestock products. All agri-environmental (for 32 measures) and less-favoured payments, CAP premiums, prices and production costs of the commodities listed above are simultaneously accounted for in up to 40 regional and structural (i.e., alpine farming zones) production units. A set of detailed feed and fertilizer balances and a transport matrix assure realistic and robust production responses.

However, perfect calibration and aggregation in agricultural sector models is still not possible, even if administrative data on single farm observations of activity levels and financial information of government transfers were available. An assumption must be made about the missing cost information and the curvature of the cost function outside the observed average cost. Currently, the choice of the functional form is arbitrary, which affects the model response to policy and price changes. Also, parameters of the A matrix and the resource endowment vector (B) should be included in a more complete calibration process. Furthermore, it is necessary to learn more about the stability of model parameters over time and spatial differences between cross-elasticities of different crops and management practices (e.g., organic and conventional produced wheat and corn).

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Figure 1: Illustration of the linear PMP approximation approach

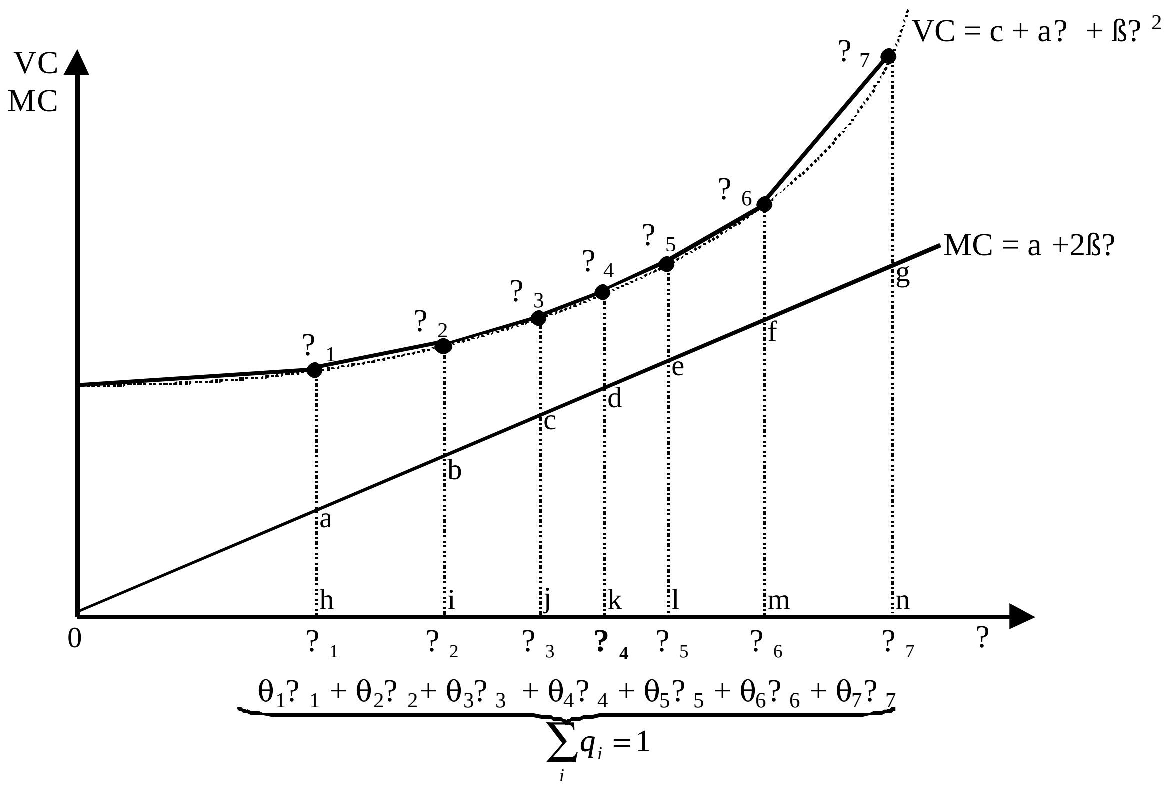

Table 1: Example of convex combinations of crop mixes (in 1,000 ha)

		mix_1	mix_2	mix_3	mix_4	mix_5	mix_6	mix_m
wheat	w/o cover crops	30	25	20	22	15	24	...
	w cover crops					12		
barley		30	32	25	28	23	28	...
corn		20	25	30	22	28	24	...
potatoes		8	10	15	12	11	14	...
set aside		12	8	10	16	11	10	...
		$?_1?_1$	$?_2?_2$	$?_3?_3$	$?_4?_4$	$?_5?_5$	$?_6?_6$	$?_m?_m$

$$\sum_m k_m = 1$$

Source: own construction.