Robust Estimation of the Spectral Density Function with Applications to the Analysis of Heart Rate Variability

Bernhard Spangl
University of Natural Resources and Applied Life Sciences
Department of Landscape, Spatial and Infrastructure Sciences
Institute for Applied Statistics and Computing
Gregor Mendel Str. 33
A–1180 Vienna, Austria
bernhard.spangl@boku.ac.at

Rudolf Dutter
Vienna University of Technology
Department of Statistics and Probability Theory
Wiedner Hauptstr. 8
A–1040 Vienna, Austria
R.Dutter@tuwien.ac.at

Motivation

Spectral estimation in

- geophysics
- signal processing
- medicine → analysis of heart rate variability (HRV) as noninvasive method

Data Model with Additive Outliers (AO)

The process $\{y_t : t \in \mathbb{Z}\}$ is said to have additive outliers (AO) if it is defined by

$$y_t = x_t + v_t$$

where x_t is an ARMA(p, q) process and the v_t are independent, identically distributed (iid) with $F_v = (1 - \varepsilon)\delta_0 + \varepsilon H$ where δ_0 is the degenerated distribution having all its mass at the origin and H is a heavy-tailed symmetric distribution with mean 0 and variance σ_H^2.

Robust Filter-cleaner

Let $\{y_k, k = 1, \dots, N\}$ denote the observed values of a second-order stationary process with mean zero. The filter-cleaner algorithm (cf. Martin and Thomson, 1982) relies on the AR(p) approximation of the underlying process x_t, represented in the following state-space form with $t = p + 1, \dots, N$:

$$\boldsymbol{X}_t = \boldsymbol{\Phi}\boldsymbol{X}_{t-1} + \boldsymbol{U}_t \;,$$

where

$$\begin{aligned} \boldsymbol{X}_t &= (x_t, x_{t-1}, \dots, x_{t-p+1})^\top \;, \\ \boldsymbol{U}_t &= (\varepsilon_t, 0, \dots, 0)^\top \;, \end{aligned}$$

with $\boldsymbol{\Phi} = \begin{pmatrix} \phi_{1,p} & \cdots & \phi_{p-1,p} & \phi_{p,p} \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$, $\operatorname{cov}(\boldsymbol{U}_t) = \boldsymbol{Q} = \begin{pmatrix} \sigma^2_{\varepsilon,p} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ and

$$y_t = x_t + v_t = (1, 0, \dots, 0)\boldsymbol{X}_t + v_t \quad \text{with} \quad \operatorname{var}(v_t) = \sigma_0^2 \;.$$

The algorithm computes robust estimates $\widehat{\boldsymbol{X}}_t$ of the unobservable $\boldsymbol{X}_t$ according to the following recursion:

$$\widehat{\boldsymbol{X}}_t = \boldsymbol{\Phi}\widehat{\boldsymbol{X}}_{t-1} + \frac{\boldsymbol{m}_t}{s_t^2}\, s_t\, \psi\left(\frac{y_t - \hat{y}_t^{t-1}}{s_t}\right)$$

with $\boldsymbol{m}_t$ being the first column of $\boldsymbol{M}_t$, which is computed recursively as

$$\begin{aligned} \boldsymbol{M}_{t+1} &= \boldsymbol{\Phi}\boldsymbol{P}_t\boldsymbol{\Phi}^\top + \boldsymbol{Q} \\ \boldsymbol{P}_t &= \boldsymbol{M}_t - w\left(\frac{y_t - \hat{y}_t^{t-1}}{s_t}\right)\frac{\boldsymbol{m}_t\boldsymbol{m}_t^\top}{s_t^2} \;. \end{aligned}$$

The weight function $w(r) = \psi(r)/r$ where ψ stands for some robustifying function. The scale s_t is defined by $s_t^2 = m_{11,t}$ and $\hat{y}_t^{t-1}$ denotes a robust one-step prediction of y_t based on $\boldsymbol{Y}^{t-1} = (y_1, \dots, y_{t-1})^\top$, and is given by

$$\hat{y}_t^{t-1} = (\boldsymbol{\Phi}\widehat{\boldsymbol{X}}_{t-1})_1 \;.$$

Finally, the cleaned process at time t results in

$$\hat{x}_t = (\widehat{\boldsymbol{X}}_t)_1 \;.$$

To use the filter-cleaner algorithm we need robust estimates $\widehat{\boldsymbol{\phi}}_p$ and $\hat{\sigma}^2_{\varepsilon,p} = s^2_{\varepsilon,p}$ of $\boldsymbol{\phi}_p = (\phi_{1,p}, \dots, \phi_{p,p})^\top$ and $\sigma^2_{\varepsilon,p}$.

Different approaches to obtain initial estimates lead to similar results, namely

- using bounded-influence autoregression (BIAR) via iteratively reweighted least squares (IWLS) (cf. Martin and Thomson, 1982)
- a highly robust autocovariance function estimator (cf. Ma and Genton, 2000) and the Yule-Walker equations
- robust autoregression using LTS- and LMS-regression

Robust Spectral Estimation via WOSA

WOSA stands for Welch's Overlapped Segment Averaging.
Let y_t, $t = 1, \dots, N$, be the observed process.

- Split the process into N_B overlapping blocks of length N_S.
- Calculate direct spectral estimates for different blocks of N_S contiguous data values

$$\widehat{S}_l^{(d)}(f) = \Delta t \left| \sum_{t=1}^{N_S} h_t y_{t+l-1} e^{-i2\pi f t \Delta t} \right|^2 , \quad l = 1\ (n)\ N - N_S + 1,$$

where $h_1, \dots, h_{N_S}$ is a data taper.

- Then the WOSA spectral estimator is defined by

$$\widehat{S}^{(WOSA)}(f) = \frac{1}{N_B} \sum_{j=0}^{N_B - 1} \widehat{S}^{(d)}_{jn+1}(f) \;.$$

A robust spectral estimator can be obtained by replacing the sample mean by an M-estimator (cf. Chave et al., 1987), i.e.,

$$\min_\theta \sum \rho\left(\frac{x_i - \theta}{s}\right) \;, \text{ or, equivalently,} \quad \sum \psi\left(\frac{x_i - \theta}{s}\right) = 0 \;,$$

where x_i corresponds to $\widehat{S}^{(d)}_{jn+1}(f)$ and $\psi(r) = \rho'(r)$. The solution $\widehat{\theta}$ is called M-estimate. Because outlier contamination can only result in a spectrum that is biased upwards, a special asymmetric ρ-function is used. It is defined by

$$\psi(r) = r \exp(-\exp(\beta(r - \beta))) \;.$$

The solution is calculated using iteratively reweighted least squares (IWLS) with proper initial values, e.g., the sample median and a corrected version of the median absolute deviation (MAD).

Simulated Data

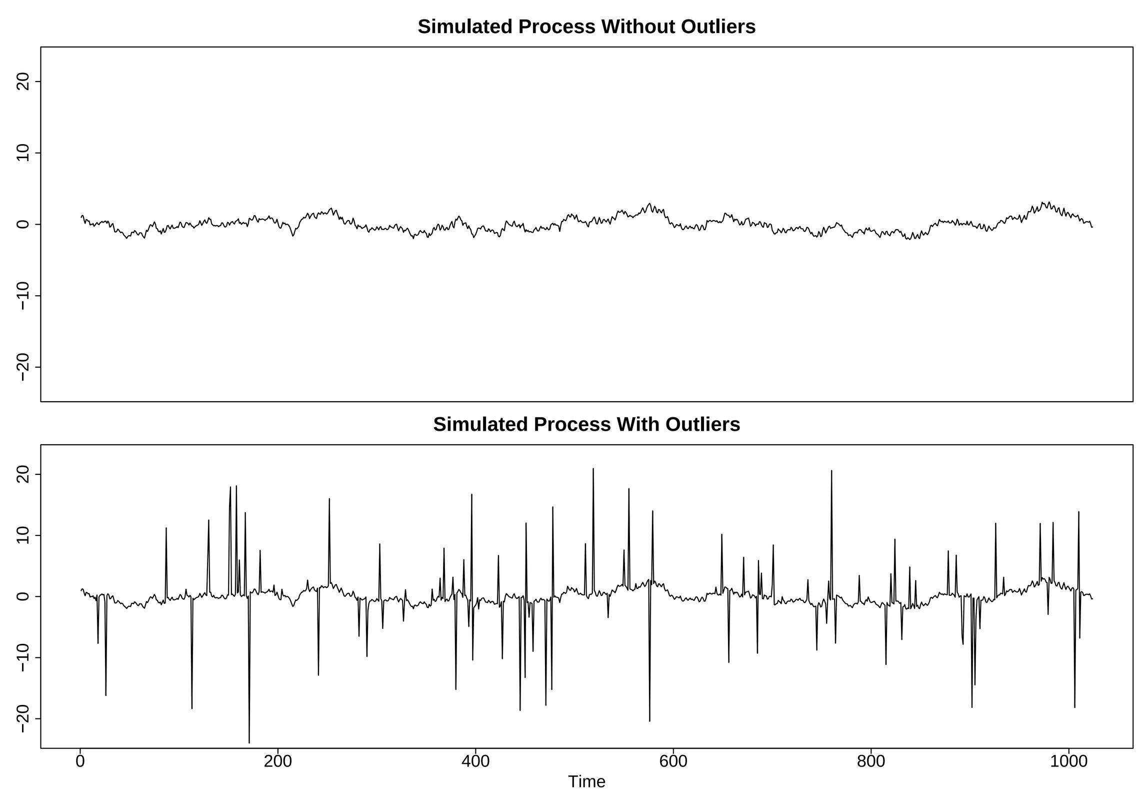
Consider the artificial process with AO composed of the following three autoregressive processes (Kleiner et al., 1979):

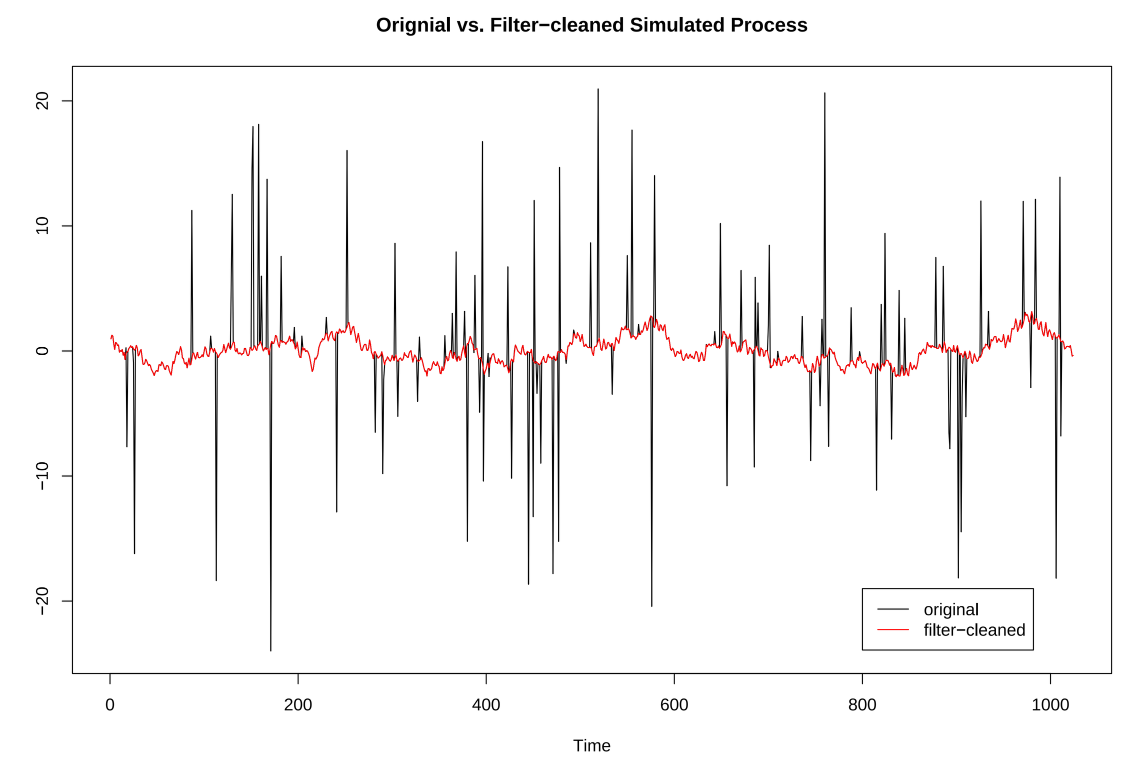
$$\begin{aligned} u_k &= 0.975u_{k-1} + \varepsilon_k \;, \\ w_k &= 0.95w_{k-1} - 0.9w_{k-2} + \eta_k \;, \\ z_k &= 0.33z_{k-1} - 0.9z_{k-2} + \zeta_k \;, \end{aligned}$$

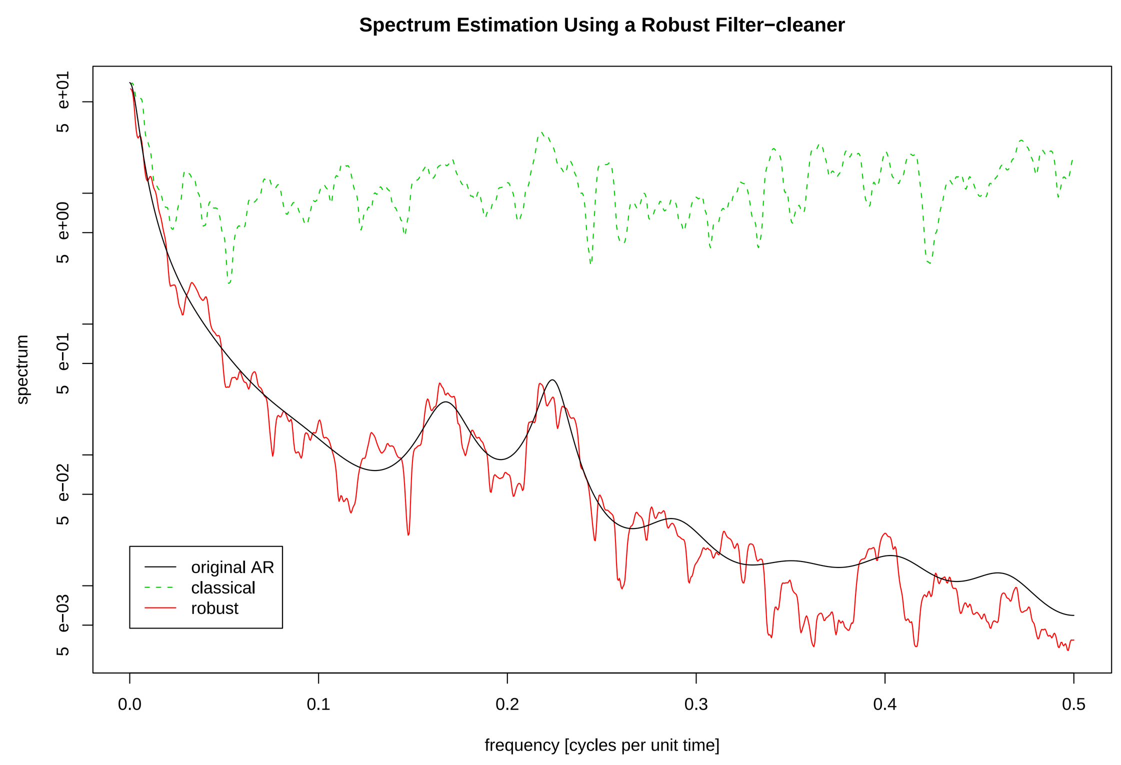
with $\varepsilon_k, \eta_k, \zeta_k \sim N(0, 1)$. u_k, w_k and z_k are standardized and

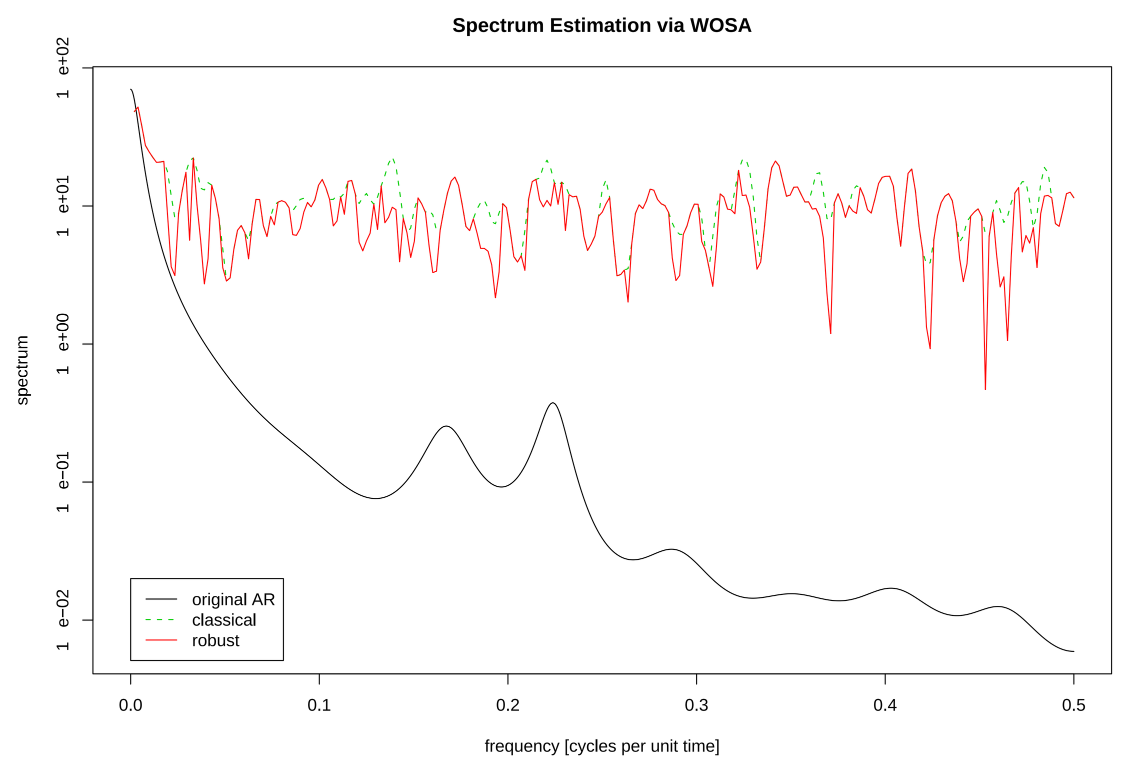
$$y_k = \sqrt{75}u_k + w_k + z_k \;, \quad k = 1, \dots, n \;,$$

is computed. y_k is standardized again. Additionally, noise from $0.9\delta_0 + 0.1N(0, 100)$ is added.







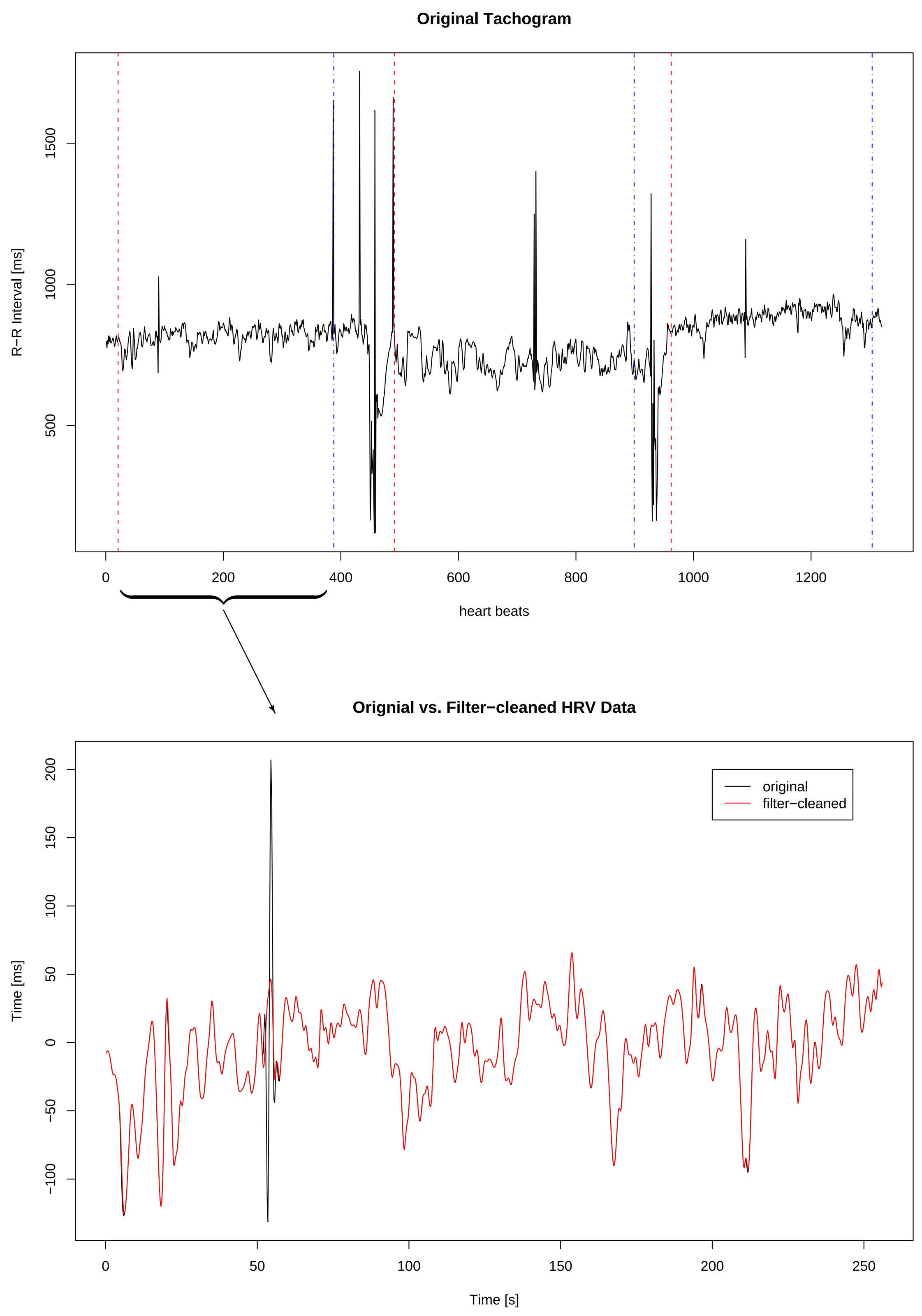


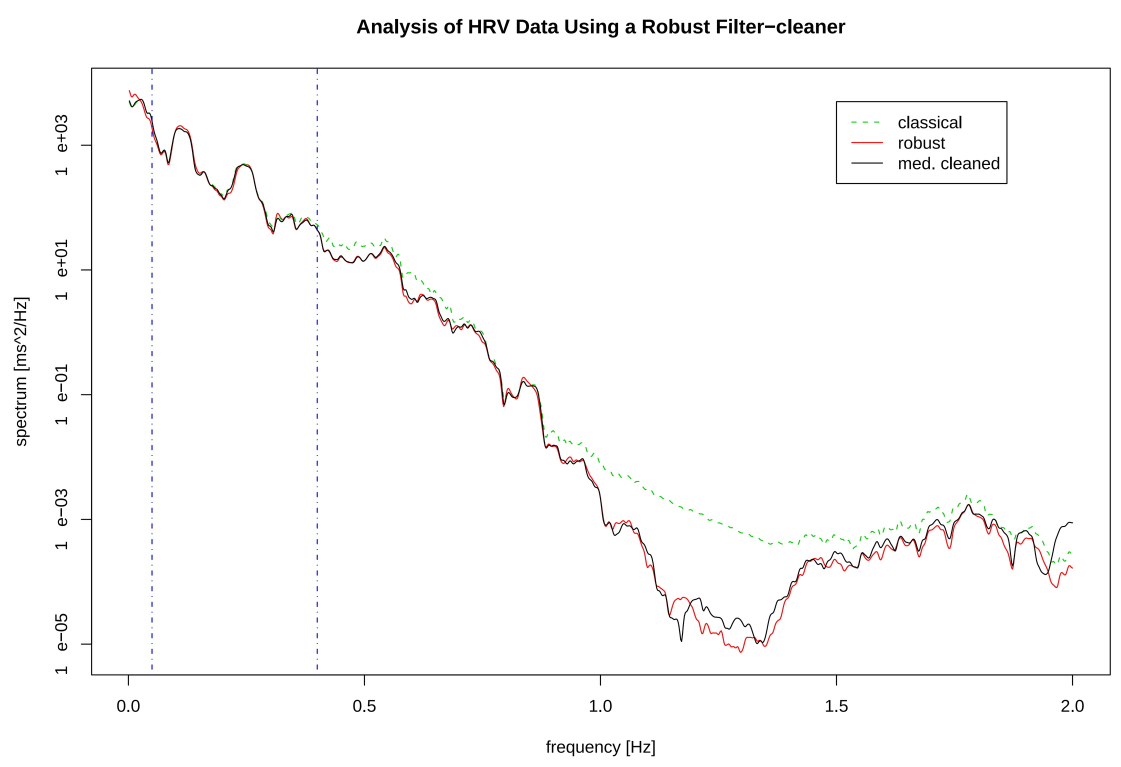
Summary & Discussion

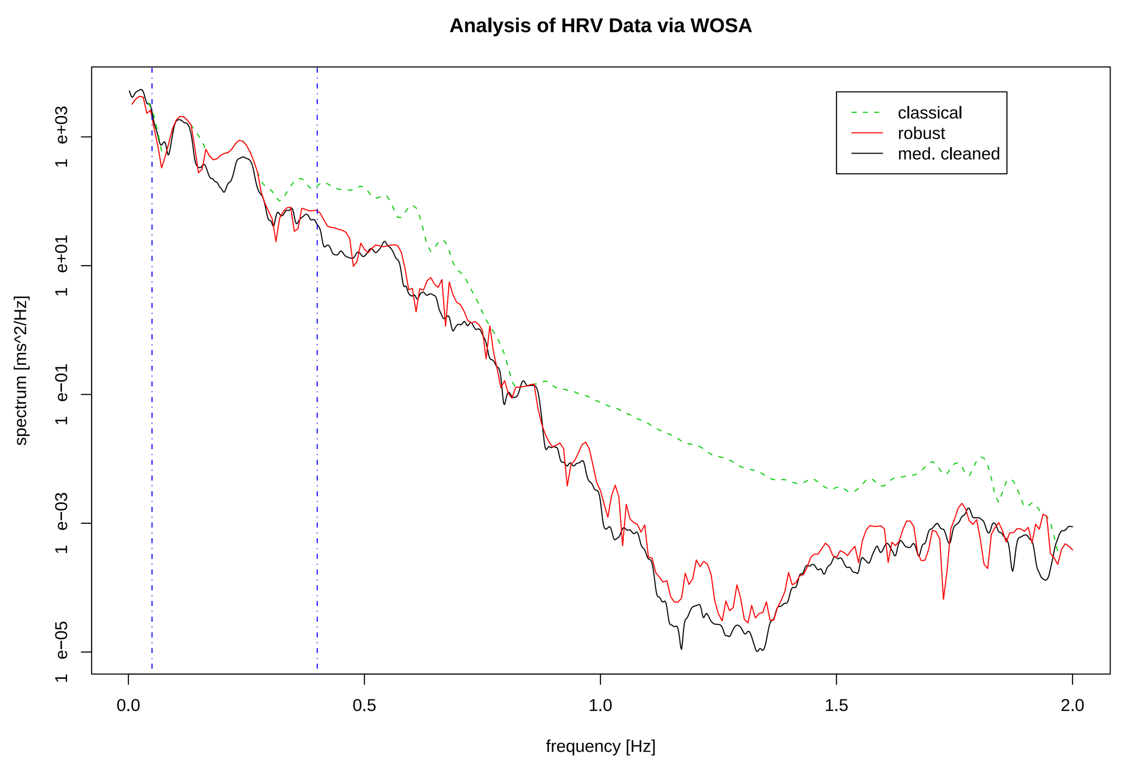
- Simulated process
  - spectral estimate using the robust filter-cleaner approach: good
  - robust WOSA spectral estimate: poor
- Analysis of HRV data
  - spectral estimate using the robust filter-cleaner approach: good
  - robust WOSA spectral estimate: good

Heart Rate Variability Data

- Real data: heart rate variability (HRV) recordings (tachogram of 1321 successive heart beats)
- provided by J. Pumprla and K. Howorka, Department of Biomedical Engineering and Physics, General Hospital of Vienna







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