Robust Spectral Estimation with Applications

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Contents

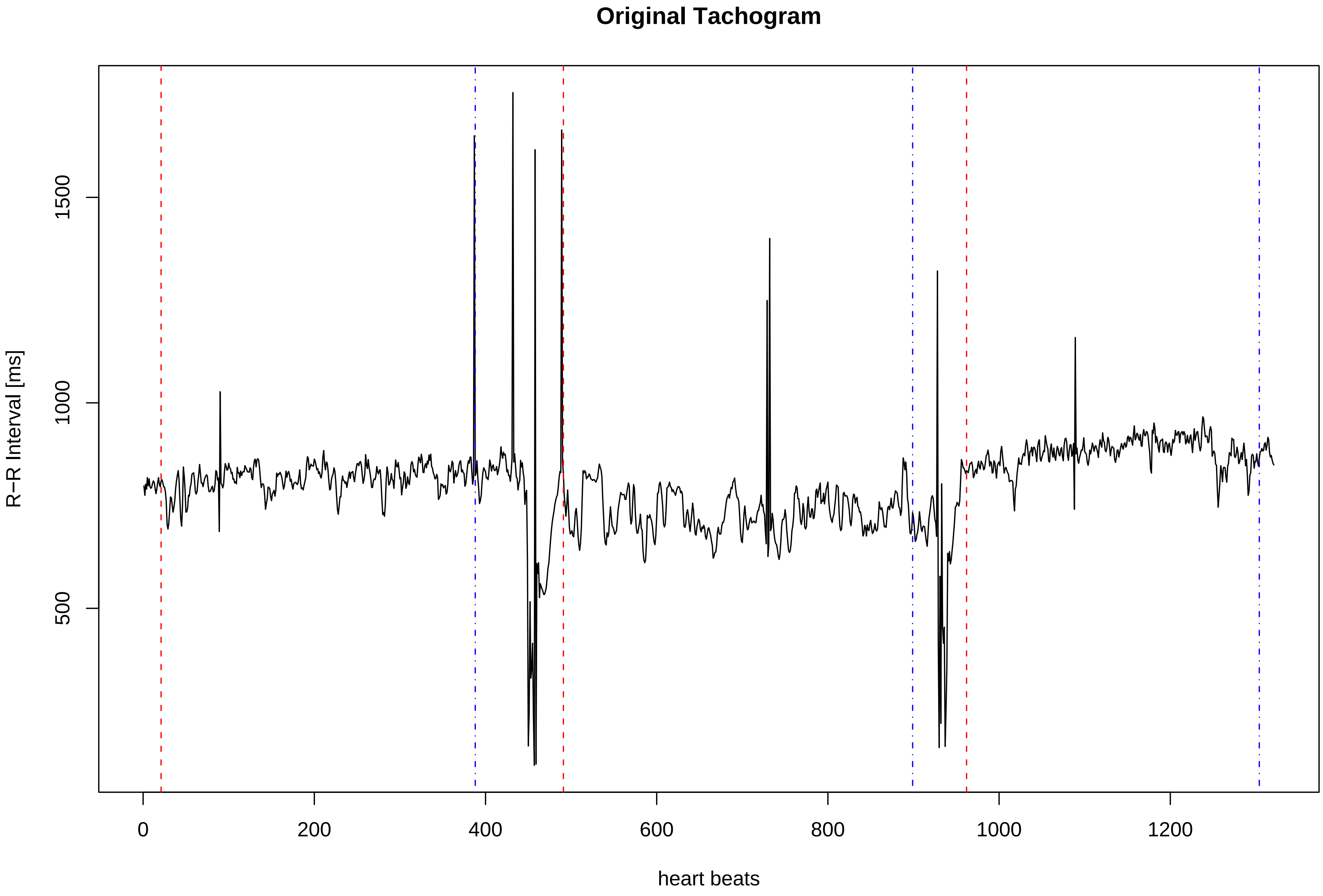
Applications of Spectral Estimation

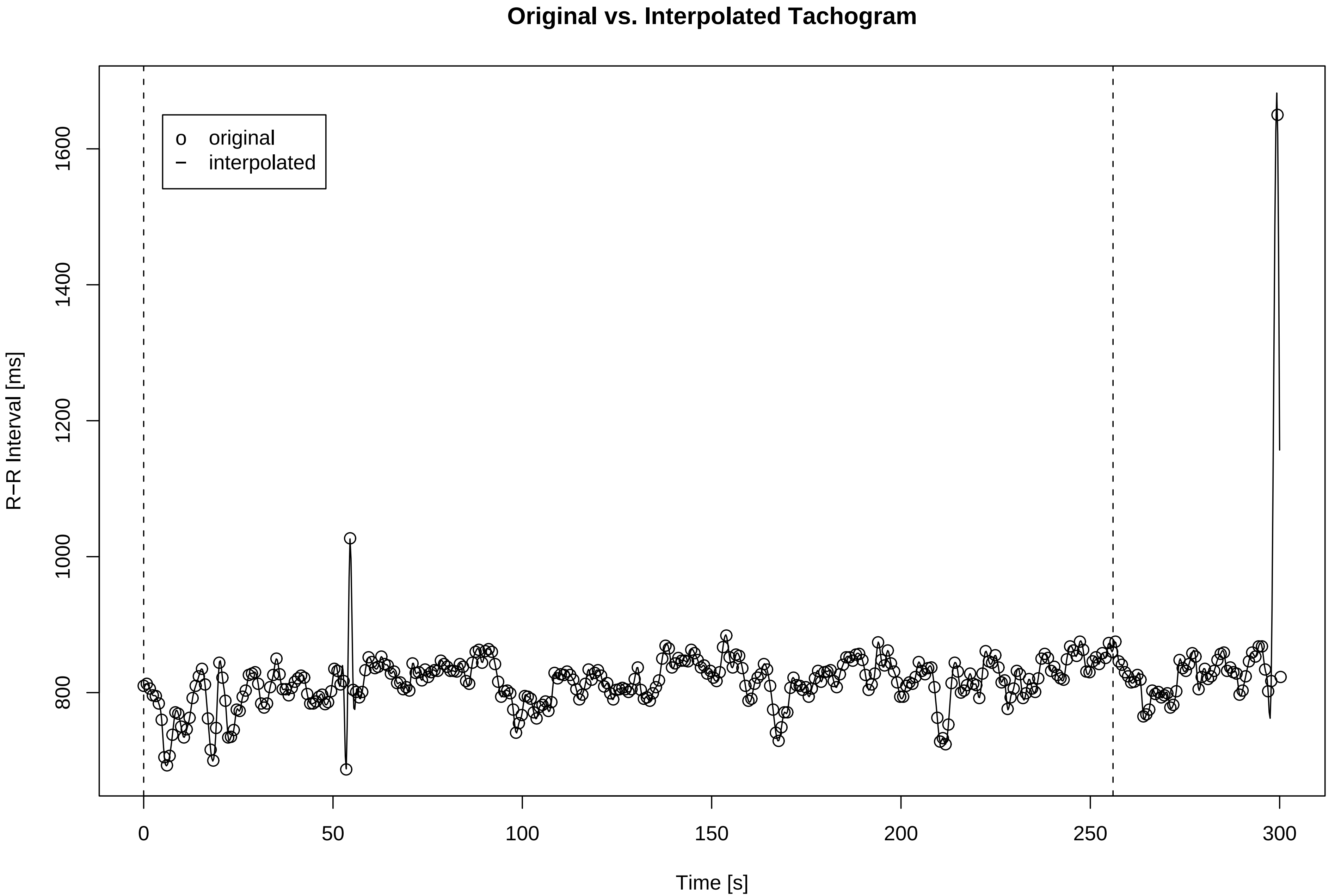
- geophysics
- signal processing
- medicine $\longrightarrow$ analysis of heart rate variability (HRV)

Heart Rate Variability Data

- data: heart rate variability (HRV) recordings
  (tachogram of 1321 successive heart beats)

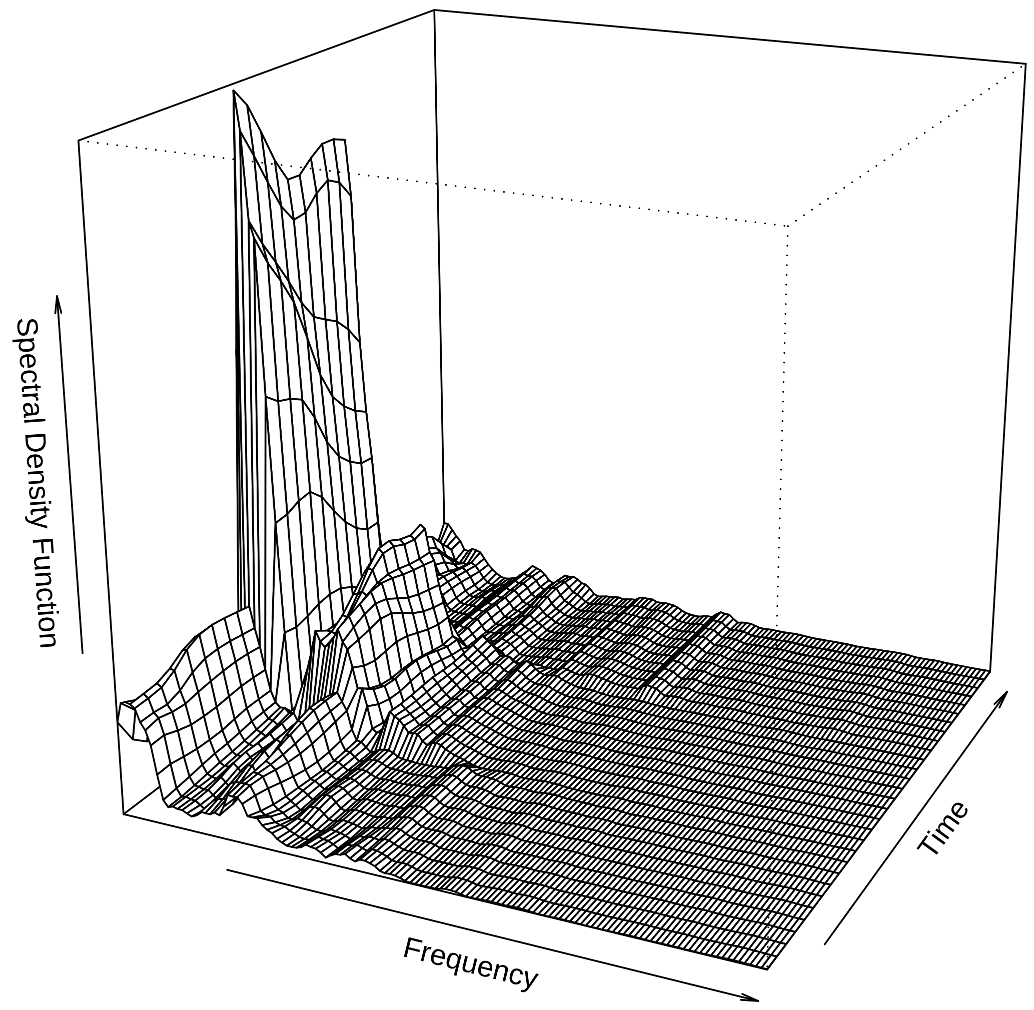
- provided by J. Pumprla and K. Howorka,
  Department of Biomedical Engineering and Physics,
  General Hospital of Vienna


Original Tachogram
R−R Interval [ms]
1500
1000
500
0
200
400
600
800
1000
1200
heart beats



Original vs. Interpolated Tachogram
o original
− interpolated
R−R Interval [ms]
800
1000
1200
1400
1600
0
50
100
150
200
250
300
Time [s]

Short−term Analysis of HRV Data



Maximum: 37.979 *10^3 [ms^2/Hz]

Artificial Example

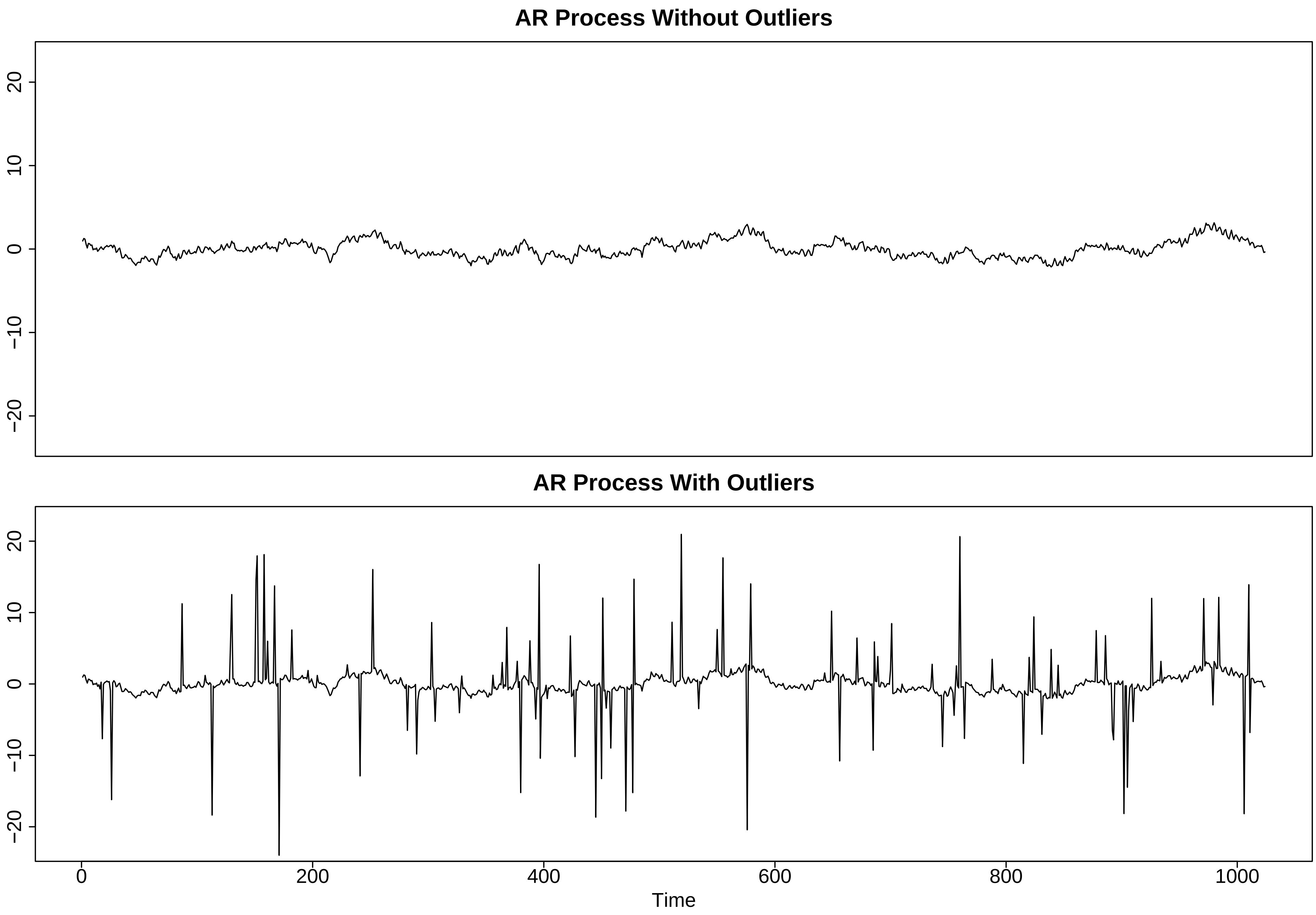
This example is a artificial process with AO composed of the following three autoregressive processes:

$$
\begin{aligned}
u_k &= 0.975 u_{k-1} + \varepsilon_k \ , \\
w_k &= 0.95 w_{k-1} - 0.9 w_{k-2} + \eta_k \ , \\
z_k &= 0.33 z_{k-1} - 0.9 z_{k-2} + \zeta_k \ ,
\end{aligned}
$$

with $\varepsilon_k, \eta_k, \zeta_k \sim N(0,1)$. We standardize u_k, w_k and z_k, and compute

$$
y_k = \sqrt{75} u_k + w_k + z_k \ , \quad k = 1, \ldots, n \ ,
$$

and then standardize y_k again. Additionally we add some noise from $0.9\delta_0 + 0.1 N(0, 10)$.


AR Process Without Outliers
20
10
0
−10
−20
AR Process With Outliers
20
10
0
−10
−20
0
200
400
600
800
1000
Time

Basics

- Second-order Stationarity
- Additive Outliers
- Spectral Representation Theorem
- Non-parametric Estimation
- Parametric Estimation
- Estimates Based on Prewhitening

Additive Outliers (AO)

We say that the process $\{y_t : t \in \mathbb{Z}\}$ has additive outliers (AO) if it is defined by

$$y_t = x_t + v_t \tag{1}$$

where x_t is an ARMA(p, q) process and the v_t are independent identically distributed (iid) with common distribution $F_v = (1 - \varepsilon)\delta_0 + \varepsilon H$ where δ_0 is the degenerate distribution having all its mass at the origin and H is a heavy-tailed symmetric distribution with mean 0 and variance σ^2. Hence, the ARMA(p, q) process x_t is observed with probability $1 - \varepsilon$ whereas the ARMA(p, q) process plus an error v_t is observed with probability ε. We shall also assume that x_t and v_t are independent.

Estimates Based on Prewhitening

Let $\{x_k, k = 1, \ldots, N\}$ denote the observed values of a second-order stationary process with mean zero. Kleiner et al. (1979) prefer the following autoregression prewhitened spectral density estimate which was originally suggested by Blackman & Tukey (1958):

$$\widehat{S}(f) = \frac{\widehat{S}_r^{(lw)}(f)}{|\widehat{H}_p(f)|^2} \,, \tag{2}$$

where $\widehat{S}_r^{(lw)}(f)$ is a lag window spectral estimate of the prediction residuals $r_k = x_k - \sum_{j=1}^{p} \widehat{\phi}_{j,p} x_{k-j}$, $k = p+1, \ldots, N$ and

$$\widehat{H}_p(f) = 1 - \sum_{j=1}^{p} \widehat{\phi}_{j,p} e^{-i2\pi f j \Delta t} \,. \tag{3}$$

One proposal is to robustify both the numerator $\widehat{S}_r^{(lw)}(f)$ and the denominator $\widehat{H}_p(f)$ in (2) which leads to the robust Filter-cleaner approach suggested by Martin & Thomson (1982).

Robust Filter-cleaner (Overview)

Let $y_t = x_t + v_t$ denote the observed process.

- Order selection of the underlying AR(p) process
- Calculating the robust estimates $\widehat{\phi}_p = (\widehat{\phi}_{1,p}, \ldots, \widehat{\phi}_{p,p})^\top$ and $\widehat{\sigma}^2_{\varepsilon,p}$
- Robust Filter-Cleaning of the process y_t
- Calculating the prewhitened spectral estimate $\widehat{S}(f)$

Robust Filter-cleaner (Part I)

Let $\{y_k, k = 1, \ldots, N\}$ denote the observed values of a second-order stationary process with mean zero. The filter-cleaner algorithm rely on the AR(p) approximation of the underlying process x_t, represented in the following state-space form with $t = p+1, \ldots, N$:

$$\boldsymbol{X}_t = \boldsymbol{\Phi} \boldsymbol{X}_{t-1} + \boldsymbol{U}_t \ , \tag{4}$$

where

$$\boldsymbol{X}_t = (x_t, x_{t-1}, \ldots, x_{t-p+1})^\top \ , \tag{5}$$

$$\boldsymbol{U}_t = (\varepsilon_t, 0, \ldots, 0)^\top \ , \tag{6}$$

$$\text{with } \boldsymbol{\Phi} = \begin{pmatrix} \phi_{1,p} & \phi_{2,p} & \cdots & \phi_{p,p} \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} , \quad \boldsymbol{Q} = \begin{pmatrix} \sigma^2_{\varepsilon,p} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \text{ and} \tag{7}$$

$$y_t = x_t + v_t = (1, 0, \ldots, 0)\boldsymbol{X}_t + v_t \quad \text{with} \quad \boldsymbol{R} = \text{var}(v_t) = \sigma_0^2 \ . \tag{8}$$

Robust Filter-cleaner (Part II)

The algorithm computes robust estimates $\widehat{\boldsymbol{X}}_t$ of the vector $\boldsymbol{X}_t$ according to the following recursion:

$$\widehat{\boldsymbol{X}}_t = \boldsymbol{\Phi}\widehat{\boldsymbol{X}}_{t-1} + \frac{\boldsymbol{m}_t}{s_t^2}\ s_t\ \psi\left(\frac{y_t - \widehat{y}_t^{t-1}}{s_t}\right) \tag{9}$$

with $\boldsymbol{m}_t$ being the first column of $\boldsymbol{M}_t$, which is computed recursively as

$$\boldsymbol{M}_{t+1} \; = \; \boldsymbol{\Phi}\boldsymbol{P}_t\boldsymbol{\Phi}^\top + \boldsymbol{Q} \tag{10}$$

$$\boldsymbol{P}_t \; = \; \boldsymbol{M}_t - w\left(\frac{y_t - \widehat{y}_t^{t-1}}{s_t}\right)\frac{\boldsymbol{m}_t\boldsymbol{m}_t^\top}{s_t^2}\ . \tag{11}$$

The scale s_t is defined by $s_t^2 = m_{11,t}$ and $\widehat{y}_t^{t-1}$ denotes a robust one-step prediction of y_t based on $\boldsymbol{Y}^{t-1} = (y_1, \ldots, y_{t-1})^\top$, and is given by

$$\widehat{y}_t^{t-1} = (\boldsymbol{\Phi}\widehat{\boldsymbol{X}}_{t-1})_1\ . \tag{12}$$

Finally, the cleaned process at time t is

$$\widehat{x}_t = (\widehat{\boldsymbol{X}}_t)_1\ . \tag{13}$$

Robust Filter-cleaner (Part III)

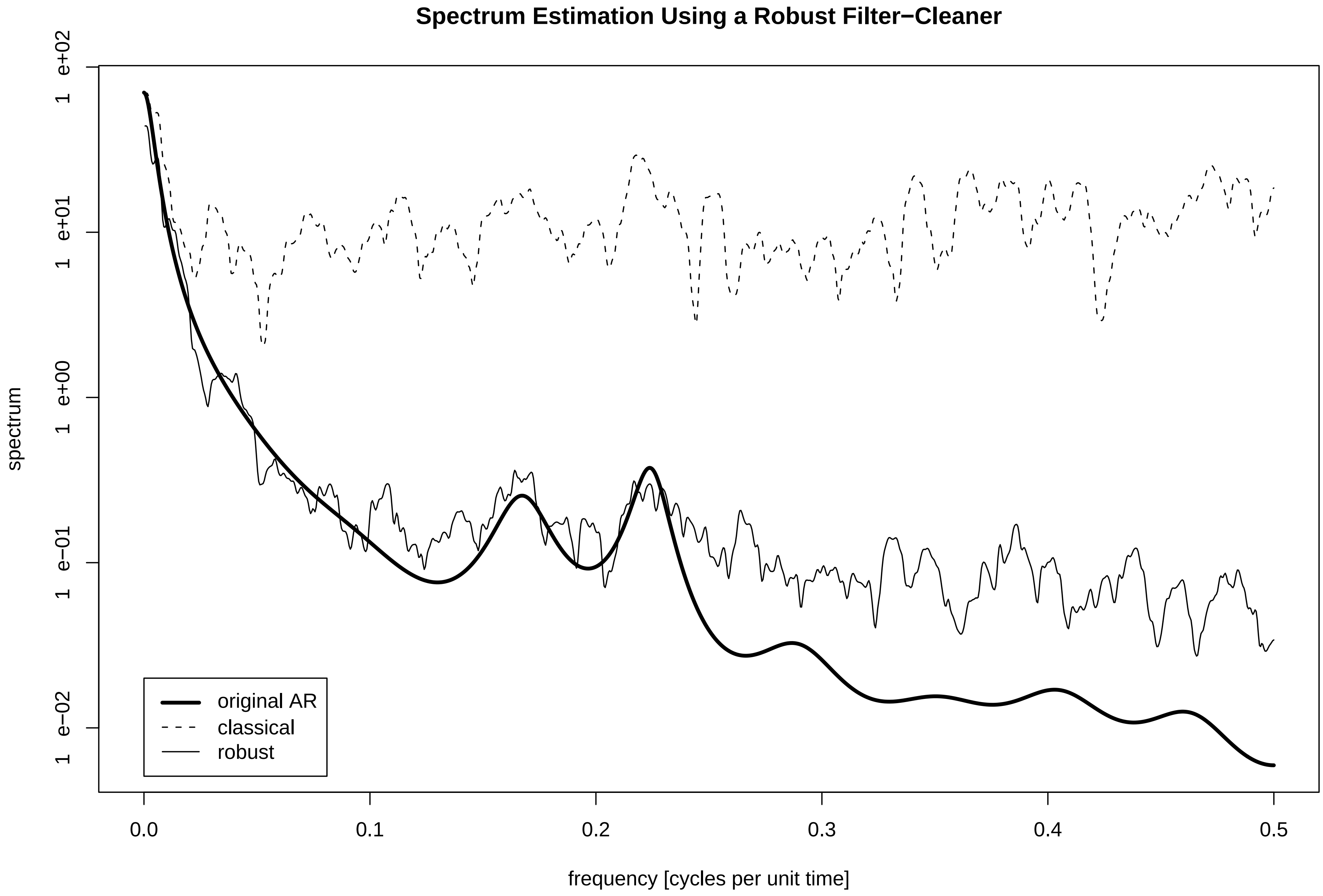
To use the filter-cleaner algorithm we need robust estimates $\widehat{\phi}_p$ and $\hat{\sigma}^2_{\varepsilon,p} = s^2_{\varepsilon,p}$ of $\phi_p = (\phi_{1,p}, \ldots, \phi_{p,p})^\top$ and $\sigma^2_{\varepsilon,p}$.

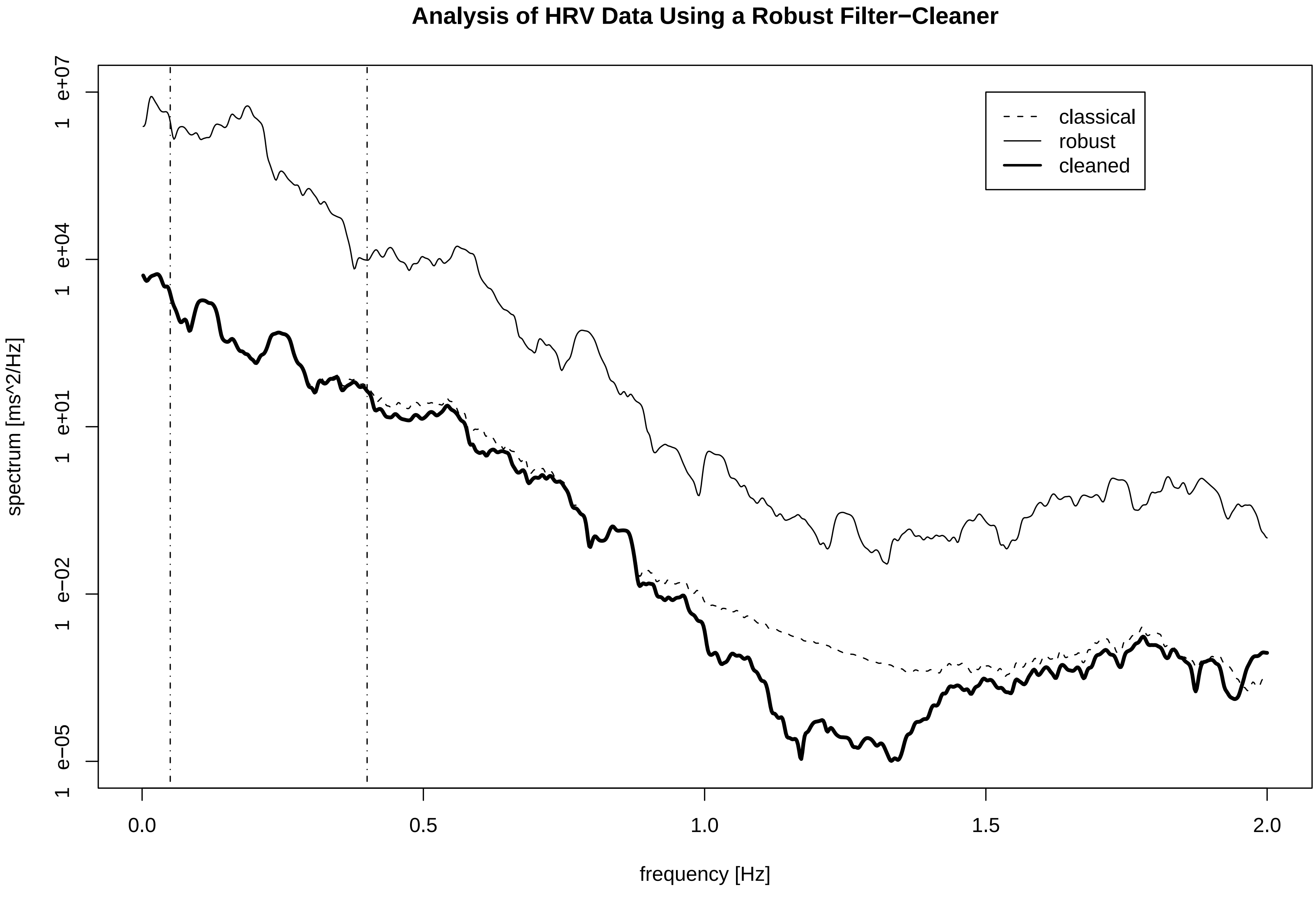
Until now, we have tried different approaches to obtain initial estimates leading to similar results

- using bounded-influence autoregression (BIAR) via iterated re-weighted least squares (IWLS) (cf. Martin & Thomson, 1982) or

- a highly robust autocovariance function estimator (cf. Ma & Genton, 2000) and the Yule-Walker equations or

- robust autoregression using LTS- and LMS-regression.


Spectrum Estimation Using a Robust Filter−Cleaner
spectrum
1 e+02
1 e+01
1 e+00
1 e−01
1 e−02
0.0
0.1
0.2
0.3
0.4
0.5
frequency [cycles per unit time]
original AR
classical
robust



Analysis of HRV Data Using a Robust Filter−Cleaner
spectrum [ms^2/Hz]
1 e+07
1 e+04
1 e+01
1 e−02
1 e−05
0.0
0.5
1.0
1.5
2.0
frequency [Hz]
classical
robust
cleaned

Robust Spectral Estimation via WOSA (Overview)

- WOSA – Welch's Overlapped Segment Averaging
- Definition of Spectral Estimation via WOSA
- Robustification of the WOSA spectral estimator

Robust Spectral Estimation via WOSA (Part 1)

Let x_t, $t = 1, \ldots, N$, be the observed process.

- Splitting the process into N_B overlapping blocks of length N_S
- Calculating direct spectral estimates for different blocks of N_S contiguous data values

$$\widehat{S}_l^{(d)}(f) := \Delta t \left| \sum_{t=1}^{N_S} h_t x_{t+l-1} e^{-i2\pi f t \Delta t} \right|^2 , \quad 1 \le l \le N + 1 - N_S . \quad (14)$$

where $h_1, \ldots, h_{N_S}$ is a data taper.

- Then the WOSA spectral estimator is defined by

$$\widehat{S}^{(WOSA)}(f) := \frac{1}{N_B} \sum_{j=0}^{N_B - 1} \widehat{S}_{jn+1}^{(d)}(f) . \quad (15)$$

Robust Spectral Estimation via WOSA (Part 2)

A robust spectral estimator can be obtained by replacing the sample mean by an M-estimator (cf. Chave et al., 1987), i.e.,

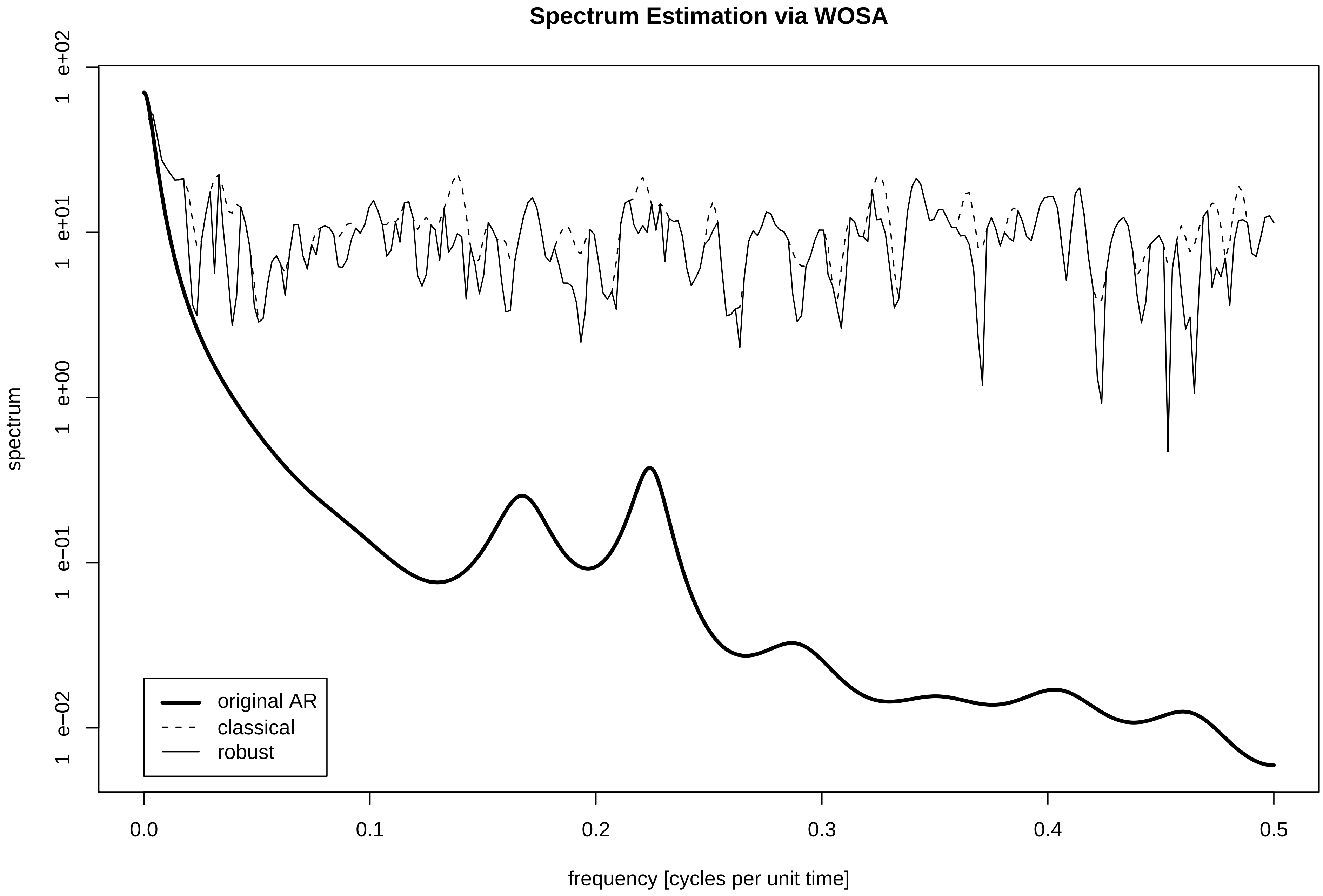
$$\min_{\theta} \sum_{i=1}^{N} \rho\left(\frac{x_i - \theta}{s}\right) \quad \Leftrightarrow \quad \sum_{i=1}^{N} \psi\left(\frac{x_i - \theta}{s}\right) = 0 \;, \tag{16}$$

where the x_i, $i = 1, \ldots, N$, are independent and $\psi(x, \theta) = \frac{\partial}{\partial\theta}\rho(x, \theta)$ is called an influence function. The solution $\widehat{\theta}$ is called an M-estimate.

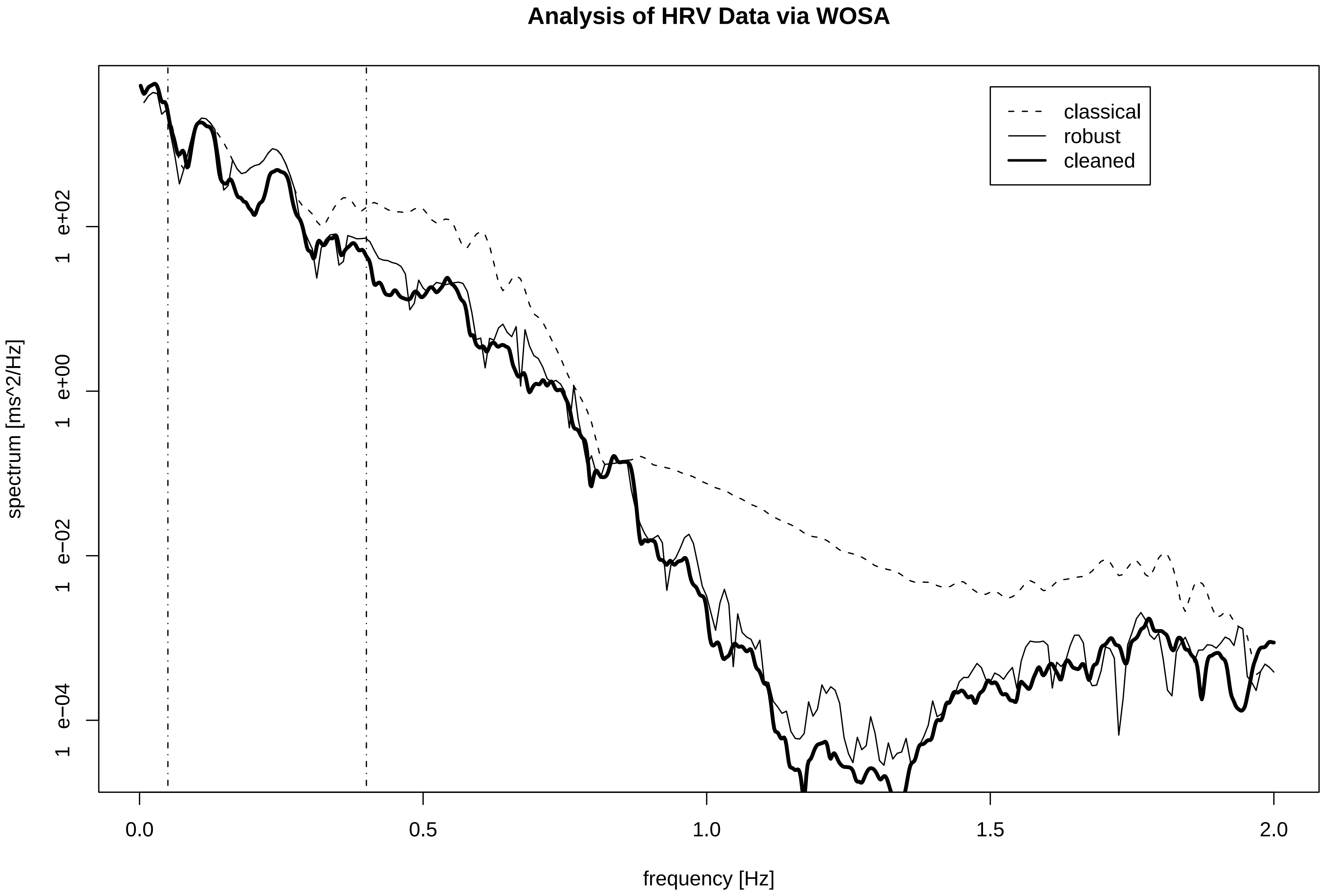
Because outlier contamination can only result in a spectrum that is biased upwards, a special asymmetric influence function is used:

$$\psi(x) = x \exp(-\exp(\beta(x - \beta))) \;. \tag{17}$$

The solution is calculated using iterated re-weighted least squares (IWLS) with proper initial values, e.g., the sample median and a corrected version of the median absolute deviation (MAD).



Spectrum Estimation via WOSA
spectrum
1 e+02
1 e+01
1 e+00
1 e−01
1 e−02
0.0
0.1
0.2
0.3
0.4
0.5
frequency [cycles per unit time]
original AR
classical
robust



Analysis of HRV Data via WOSA
classical
robust
cleaned
spectrum [ms^2/Hz]
1 e+02
1 e+00
1 e−02
1 e−04
0.0
0.5
1.0
1.5
2.0
frequency [Hz]

Summary & Discussion

- Simulated process
  - spectral estimate using the robust filter-cleaner approach: good
  - robust WOSA spectral estimate: poor

- Analysis of HRV data
  - spectral estimate using the robust filter-cleaner approach: poor
  - robust WOSA spectral estimate: good

Further Research

- Further detailed research of the algorithms
- Using a band-pass filter in the case of the analysis of HRV data
- Extension of the spectral estimators by Thomson's multitaper approach