Estimation of Power Spectra via Robust Filtering

B. Spangl and R. Dutter

BOKU - University of Natural Resources and Applied Life Sciences

and

Vienna University of Technology

(joint work with J. Pumprla and K. Howorka,
General Hospital of Vienna)

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Contents

Motivation

Analysis of Heart Rate Variability (HRV) as a non-invasive method for diabetic patients

Second-order Stationarity

Consider a time series $\{x_t : t \in \mathbb{Z}\}$ and assume that it satisfies the hypothesis of second-order stationarity:

(i) $E(x_t^2) < \infty \quad \forall\ t \in \mathbb{Z}$

(ii) $E(x_t) = \mu = \text{constant} \quad \forall\ t \in \mathbb{Z}$

(iii) $\text{cov}(x_{t+h}, x_t) = \gamma(h) \quad \forall\ t, h \in \mathbb{Z}$

where $\gamma(h)$ is the autocovariance function of x_t at lag h.

ARMA(p, q) Process

A univariate time series x_t, $t = 0, \pm 1, \pm 2, \ldots$, is said to be an ARMA($p, q$) process if x_t is stationary and

$$x_t = \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p} + w_t + \theta_1 w_{t-1} + \ldots + \theta_q w_{t-q} \ , \qquad (1)$$

with $\phi_p \neq 0$ and $\theta_q \neq 0$. The innovation process w_t in (1) is a white noise process with $\sigma_w^2 > 0$.

Additive Outliers (AO)

The process $\{y_t : t \in \mathbb{Z}\}$ is said to have additive outliers (AO) if it is defined by

$$y_t = x_t + v_t \tag{2}$$

where x_t is an ARMA(p, q) process and the v_t are independent identically distributed (iid) with common distribution $F_v = (1 - \varepsilon)\delta_0 + \varepsilon H$ where δ_0 is the degenerate distribution having all its mass at the origin and H is a heavy-tailed symmetric distribution with mean 0 and variance σ^2. Hence, the ARMA(p, q) process x_t is observed with probability $1 - \varepsilon$ whereas the ARMA(p, q) process plus an error v_t is observed with probability ε. We shall also assume that x_t and v_t are independent.

Spectral Representation Theorem

Any second-order stationary process x_t has the spectral representation

$$x_t = \int_{-1/2}^{1/2} \exp(-2\pi i t f)\, dZ(f) \tag{3}$$

where $Z(f)$, $-1/2 \leq f \leq 1/2$, has stationary uncorrelated increments. The process $Z(f)$ defines a monotone nondecreasing function $F(f)$ through

$$F(f) = E|Z(f)|^2, \quad dF(f) = E|dZ(f)|^2, \quad F(-1/2) = 0, \quad F(1/2) = \sigma^2 = \gamma(0)\,.$$

The function $F(f)$ is called the spectral distribution function. If the autocovariance function is absolutely summable, i.e., $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$, the spectral distribution function is absolutely continuous with $dF(f) = S(f)\, df$. $S(f)$ is called the *spectral density function* of the process x_t. Other commonly used terms for $S(f)$ are *spectral density*, *spectrum* or *power spectrum*.

Definition of the Power Spectrum

It follows that any process with autocovariance function $\gamma(h)$ satisfying

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty \tag{4}$$

has the representation

$$\gamma(h) = \int_{-1/2}^{1/2} \exp(2\pi i h f) S(f)\, df, \qquad h \in \mathbb{Z}\ . \tag{5}$$

Thus the autocovariance at lag h, $\gamma(h)$, $h \in \mathbb{Z}$, are the Fourier coefficients of $S(f)$ and so

$$S(f) = \sum_{h=-\infty}^{\infty} \gamma(h) \exp(-2\pi i h f)\ . \tag{6}$$

Non-parametric Estimation

The non-parametric method of estimating the spectrum is based on smoothing the periodogram. Let x_t, $t = 1, \ldots, n$, be the observed time series. Then the discrete Fourier transform of x_t

$$X(f_k) = n^{-1/2} \sum_{t=1}^{n} x_t \exp(-\frac{2\pi i t k}{n}) \ , \quad f_k = \frac{k}{n} \ , \quad k = -\left[\frac{n-1}{2}\right], \ldots, \left[\frac{n}{2}\right] \tag{7}$$

is computed using the fast Fourier transformation and the periodogram is formed by

$$I(f_k) = |X(f_k)|^2 \ . \tag{8}$$

The periodogram is smoothed to get an estimate of the power spectrum

$$\widehat{S}(f_k) = \sum_{j=-L}^{L} w_j I(f_{k+j}) \ , \quad w_j = w_{-j} \ , \quad \sum_{j=-L}^{L} w_j = 1 \ . \tag{9}$$

Parametric Estimation

We now consider basing a spectral estimator on the parameters of a p-th order autoregressive model, i.e., an AR(p) model

$$x_t - \sum_{k=1}^{p} \phi_k x_{t-k} = z_t \ , \tag{10}$$

where z_t is a white noise process with mean zero and variance σ_z^2.

Substituting the maximum likelihood or least squares estimators of the model parameters, denoted by $\widehat{\phi}_1, \ldots, \widehat{\phi}_p$ and $\widehat{\sigma}_z^2$, we obtain a parametric estimate of the power spectrum

$$\widehat{S}(f) = \frac{\widehat{\sigma}_z^2}{\left|1 - \sum_{k=1}^{p} \widehat{\phi}_k \exp(-2\pi i k f)\right|^2} \ . \tag{11}$$

Estimates Based on Robust Prewhitening and Filtering

Let $\{y_k, k = 1, \ldots, n\}$ denote the observed values of a second-order stationary process with mean zero. Kleiner et al. (1979) prefer the following autoregression prewhitened spectral density estimate which was originally suggested by Blackman & Tukey (1958):

$$\widehat{S}(f) = \frac{\widehat{S}_r(f)}{|\widehat{H}_p(f)|^2} \,, \tag{12}$$

where $\widehat{S}_r(f)$ is a smoothed periodogram estimate of the prediction residuals $r_k = y_k - \sum_{j=1}^{p} \widehat{\phi}_j y_{k-j}$, $k = p + 1, \ldots, n$ and

$$\widehat{H}_p(f) = 1 - \sum_{j=1}^{p} \widehat{\phi}_j \exp(i2\pi jf) \,. \tag{13}$$

The basic proposal is to robustify both the numerator $\widehat{S}_r(f)$ and the denominator $\widehat{H}_p(f)$ in (12).

Robust Filter-Cleaner (Part I)

The filter-cleaner algorithm rely on the AR(p) approximation of the process x_t, represented in the following state-space form with $t = p+1, \ldots, n$:

$$\boldsymbol{X}_t = \boldsymbol{\Phi} \boldsymbol{X}_{t-1} + \boldsymbol{U}_t \ , \tag{14}$$

where

$$\boldsymbol{X}_t = (x_t, x_{t-1}, \ldots, x_{t-p+1})^\top \ , \tag{15}$$

$$\boldsymbol{U}_t = (\varepsilon_t, 0, \ldots, 0)^\top \ , \tag{16}$$

$$\text{with } \boldsymbol{\Phi} = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_p \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \ , \quad \boldsymbol{Q} = \begin{pmatrix} \sigma_\varepsilon^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \text{ and} \tag{17}$$

$$y_t = x_t + v_t = (1, 0, \ldots, 0) \boldsymbol{X}_t + v_t \quad \text{with} \quad \boldsymbol{R} = \mathrm{var}(v_t) = \sigma_0^2 \ . \tag{18}$$

Robust Filter-Cleaner (Part II)

The algorithm computes robust estimates $\widehat{\boldsymbol{X}}_t$ of the vector $\boldsymbol{X}_t$ according to the following recursion:

$$\widehat{\boldsymbol{X}}_t = \boldsymbol{\Phi}\widehat{\boldsymbol{X}}_{t-1} + \frac{\boldsymbol{m}_t}{s_t^2}\ s_t\ \psi\left(\frac{y_t - \widehat{y}_t^{t-1}}{s_t}\right) \tag{19}$$

with $\boldsymbol{m}_t$ being the first column of $\boldsymbol{M}_t$, which is computed recursively as

$$\boldsymbol{M}_{t+1} = \boldsymbol{\Phi}\boldsymbol{P}_t\boldsymbol{\Phi}^\top + \boldsymbol{Q} \tag{20}$$

$$\boldsymbol{P}_t = \boldsymbol{M}_t - w\left(\frac{y_t - \widehat{y}_t^{t-1}}{s_t}\right)\frac{\boldsymbol{m}_t\boldsymbol{m}_t^\top}{s_t^2}\ . \tag{21}$$

The scale s_t is defined by $s_t^2 = m_{11,t}$ and $\widehat{y}_t^{t-1}$ denotes a robust one-step prediction of y_t based on $\boldsymbol{Y}^{t-1} = (y_1, \ldots, y_{t-1})^\top$, and is given by

$$\widehat{y}_t^{t-1} = (\boldsymbol{\Phi}\widehat{\boldsymbol{X}}_{t-1})_1\ . \tag{22}$$

Finally, the cleaned process at time t is

$$\widehat{x}_t = (\widehat{\boldsymbol{X}}_t)_1\ . \tag{23}$$

Robust Filter-Cleaner (Part III)

To use the filter-cleaner algorithm we need robust estimates $\widehat{\boldsymbol{\phi}}$ and $\hat{\sigma}_\varepsilon^2 = \hat{s}_\varepsilon^2$ of $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)^\top$ and σ_ε^2.

Until now, we have tried two different approaches to obtain initial estimates

- using bounded-influence autoregession (BIAR) via iterated re-weighted least squares (IWLS) (cf. Martin & Thomson, 1982) or

- a highly robust autocovariance function estimator (cf. Ma & Genton, 2000) and the Yule-Walker equations

$\Rightarrow$ similar results

Highly Robust Autocovariance Function

Ma & Genton (2000) suggest a highly robust autocovariance function estimator $\widehat{\gamma}_Q(h, \boldsymbol{x})$ which is defined as follows. Extract the first $n - h$ observations of $\boldsymbol{x} = (x_1, \ldots, x_n)^\top$ to produce a vector $\boldsymbol{u}$ of length $n - h$ and the last $n - h$ observations of $\boldsymbol{x}$ to produce a vector $\boldsymbol{v}$ of length $n - h$. Then:

$$\widehat{\gamma}_Q(h, \boldsymbol{x}) = \frac{1}{4}[Q^2_{n-h}(\boldsymbol{u} + \boldsymbol{v}) - Q^2_{n-h}(\boldsymbol{u} - \boldsymbol{v})] \ . \tag{24}$$

The function $Q_n(\boldsymbol{z})$ (cf. Croux & Rousseeuw, 1992) is a highly robust estimator of scale and is defined by

$$Q_n(\boldsymbol{z}) = c\{|z_i - z_j| : i < j\}_{(k)} \ , \tag{25}$$

where $\boldsymbol{z} = (z_1, \ldots, z_n)^\top$ is the sample and

$$k = \binom{h}{2} \ , \ \text{with } h = \left[\frac{n}{2}\right] + 1 \ ,$$

where [.] denotes the integer part. The factor $c = 2.2191$ is for consistency.

Artificial Data

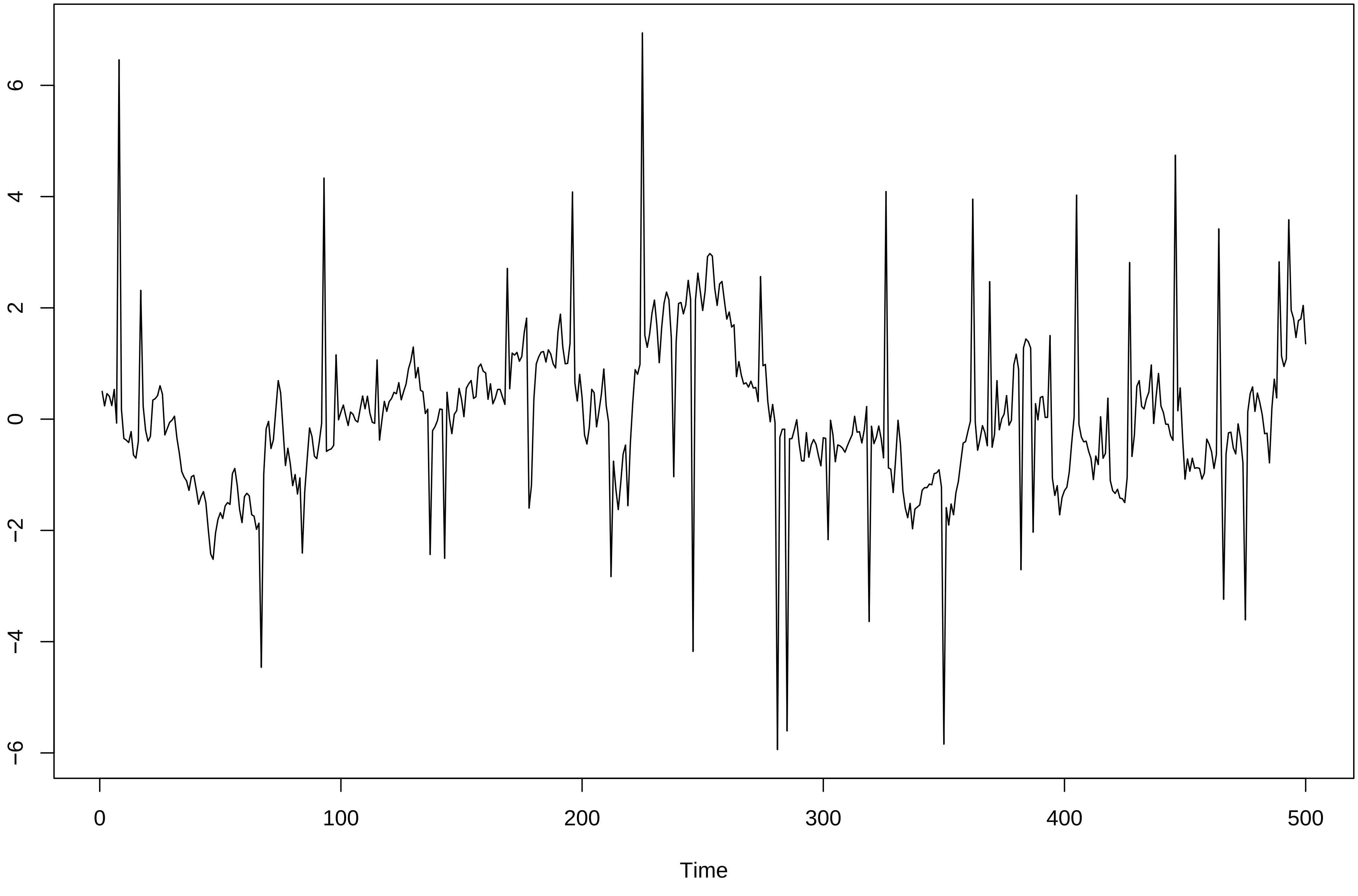
This example is a artificial process with AO composed of the following three autoregressive processes:

$$
\begin{aligned}
u_k &= 0.975 u_{k-1} + \varepsilon_k \ , \\
w_k &= 0.95 w_{k-1} - 0.9 w_{k-2} + \eta_k \ , \\
z_k &= 0.33 z_{k-1} - 0.9 z_{k-2} + \zeta_k \ ,
\end{aligned}
$$

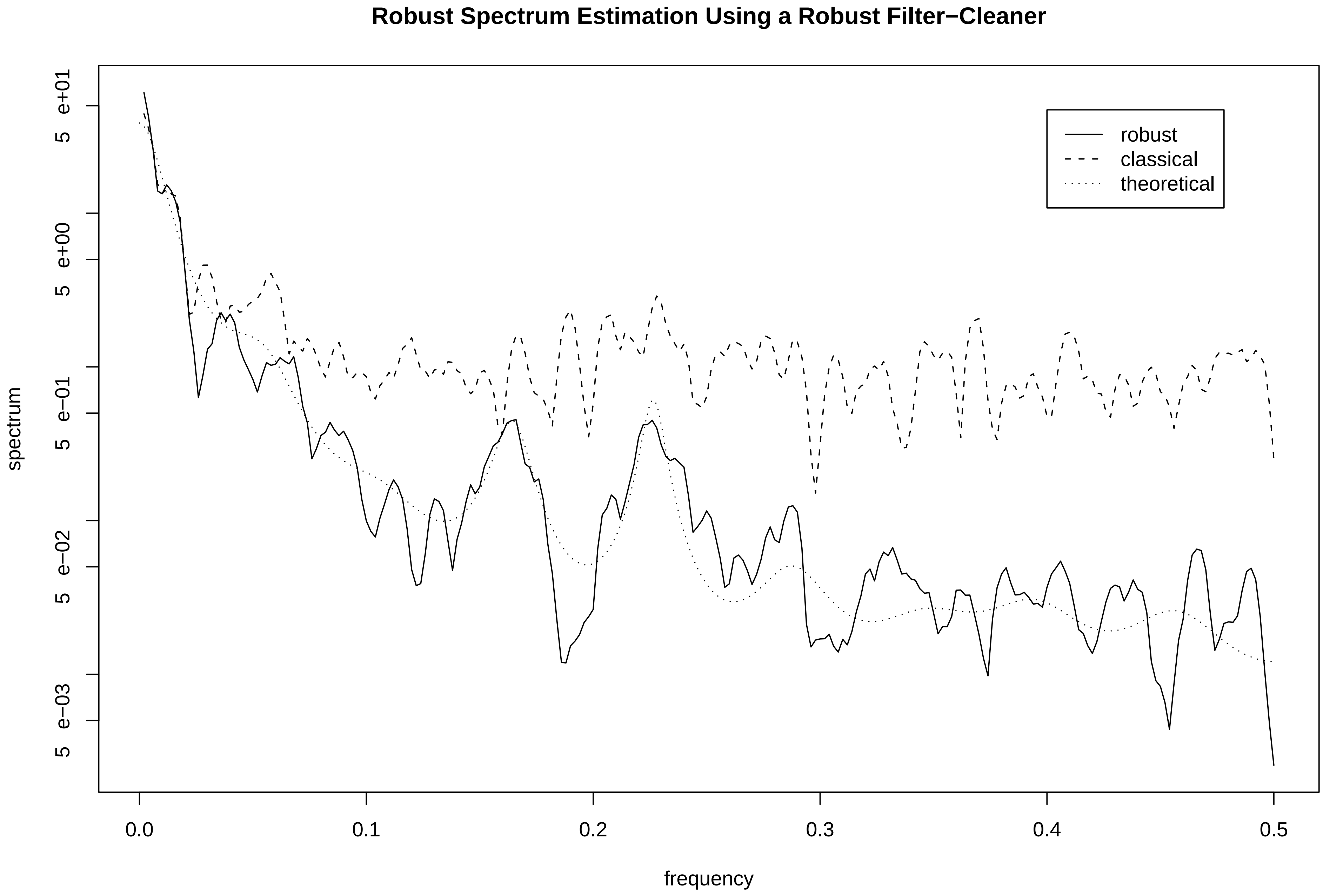
with $\varepsilon_k, \eta_k, \zeta_k \sim N(0,1)$. We standardize u_k, w_k and z_k, and compute

$$y_k = \sqrt{75} u_k + w_k + z_k \ , \quad k = 1, \ldots, n \ ,$$

and then standardize y_k again. Additionally we add some noise from $0.9\delta_0 + 0.1N(0,10)$.



6
4
2
0
−2
−4
−6
0
100
200
300
400
500
Time

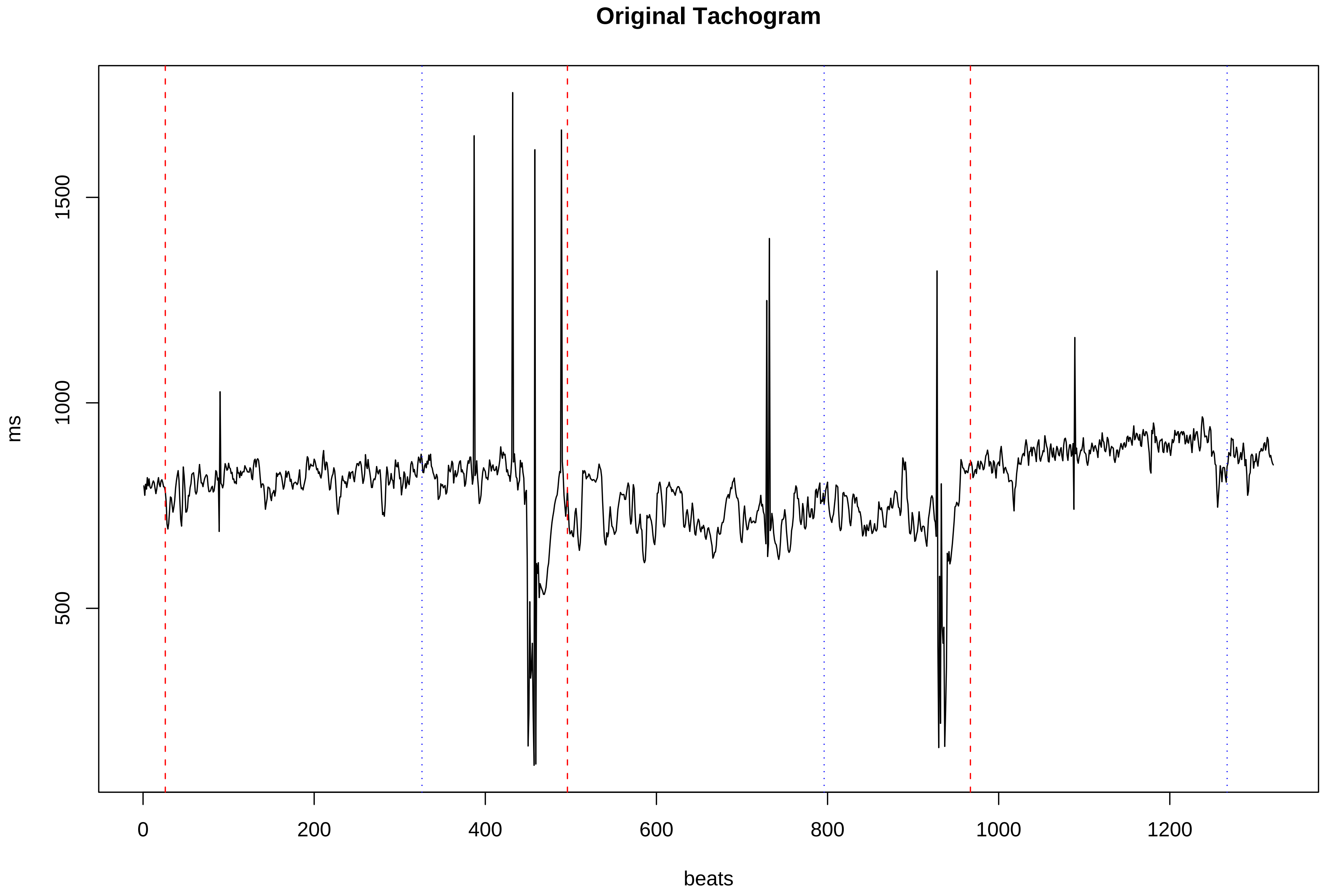


Robust Spectrum Estimation Using a Robust Filter−Cleaner
robust
classical
theoretical
spectrum
frequency
0.0
0.1
0.2
0.3
0.4
0.5
5 e−03
e−03
5 e−02
e−02
5 e−01
e−01
5 e+00
e+00
5 e+01
e+01

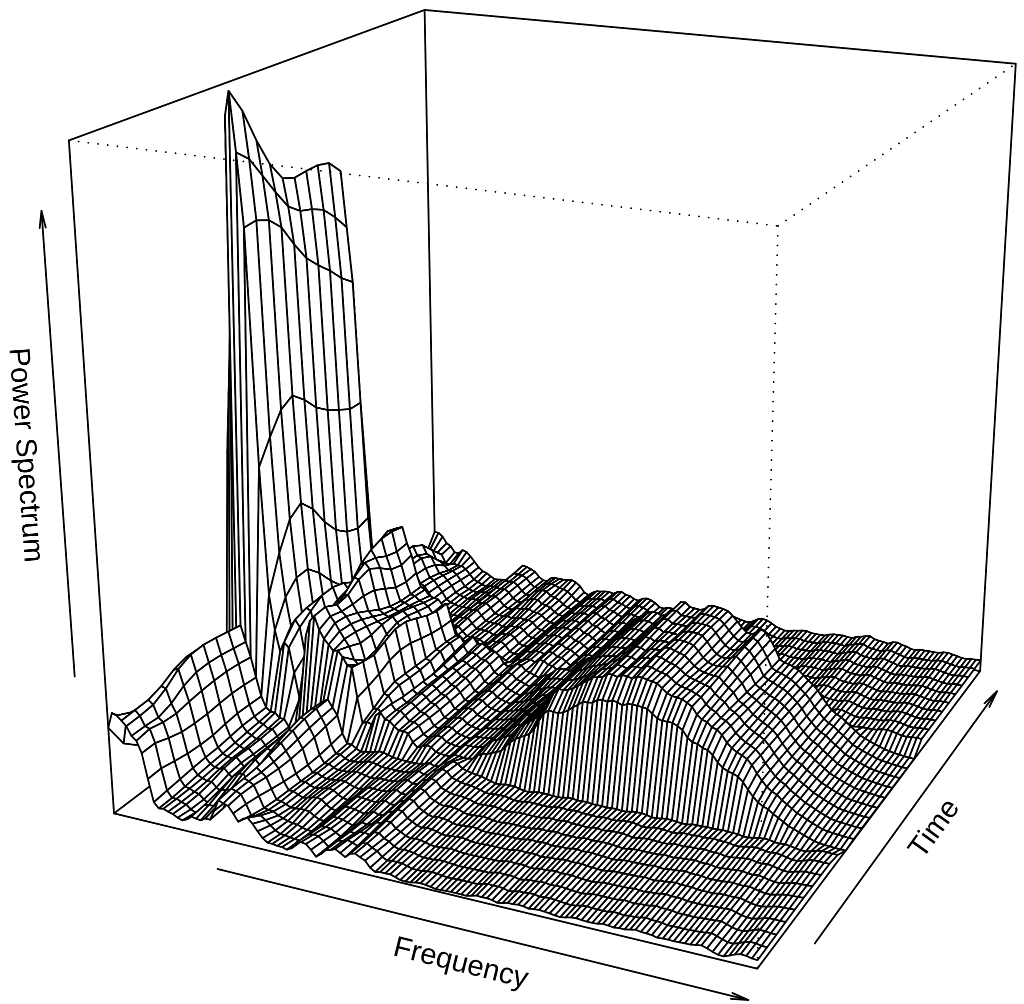
Heart Rate Variabiliy Data

- data: heart rate variability (HRV) recordings (tachogram of 1321 successive heart beats)

- provided by J. Pumprla and K. Howorka, Department of Biomedical Engineering and Physics, General Hospital of Vienna

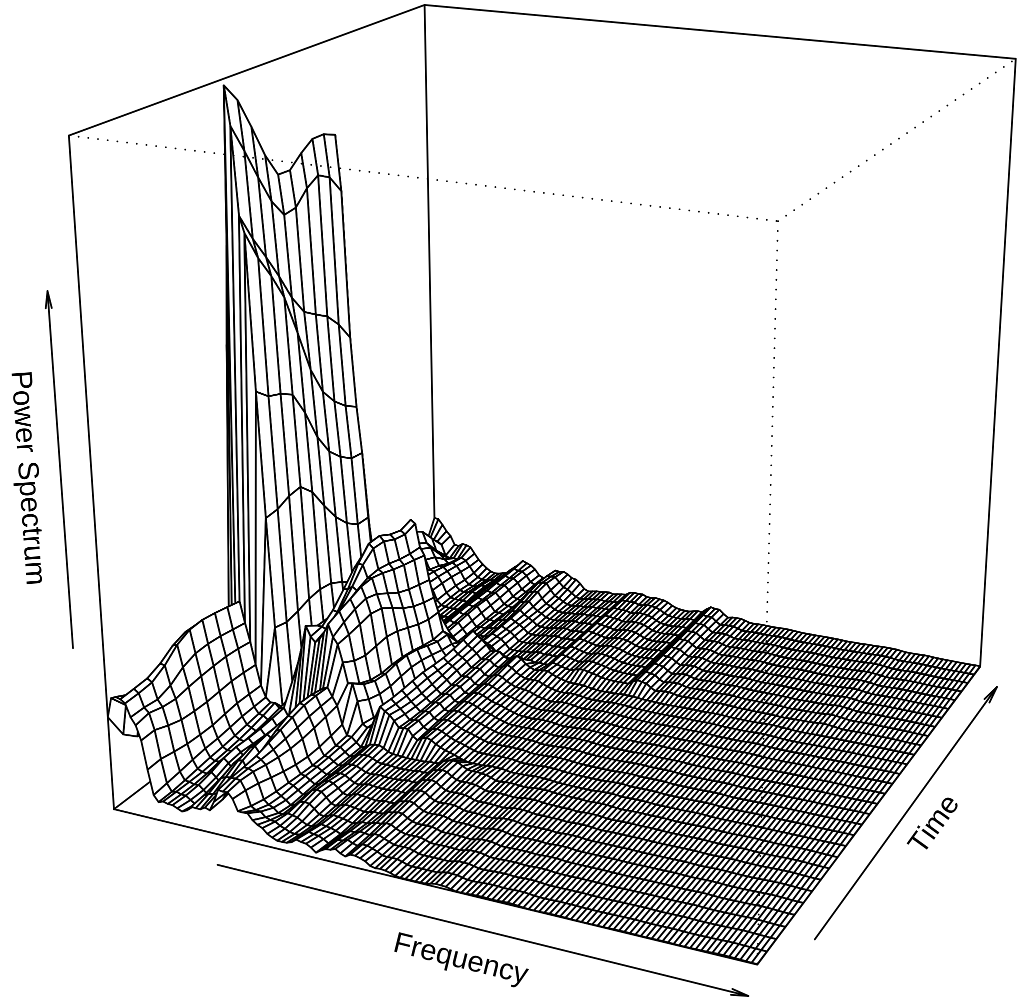

Original Tachogram
ms
1500
1000
500
0
200
400
600
800
1000
1200
beats

Classical Short−term Analysis of HRV with AO



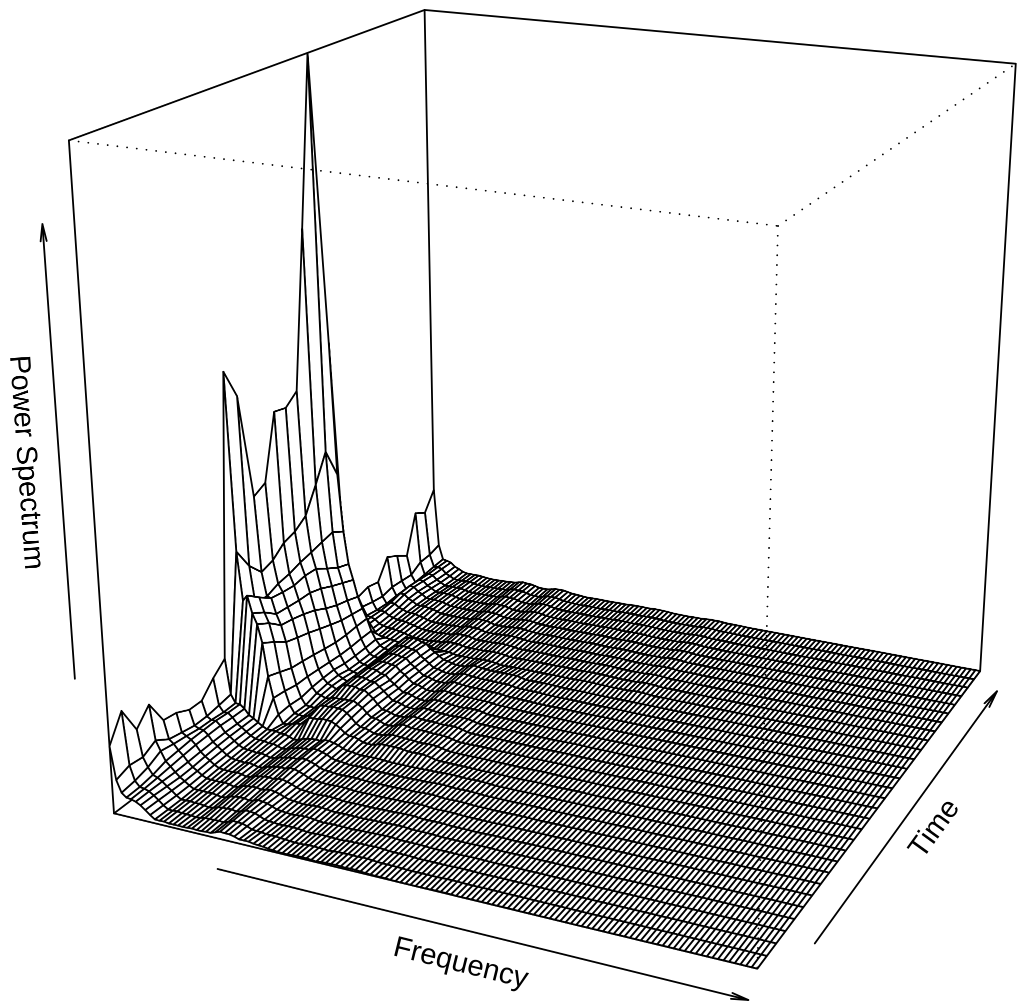
Maximum: 44.05 *10^3 [ms^2]

Classical Short−term Analysis of HRV without Outliers



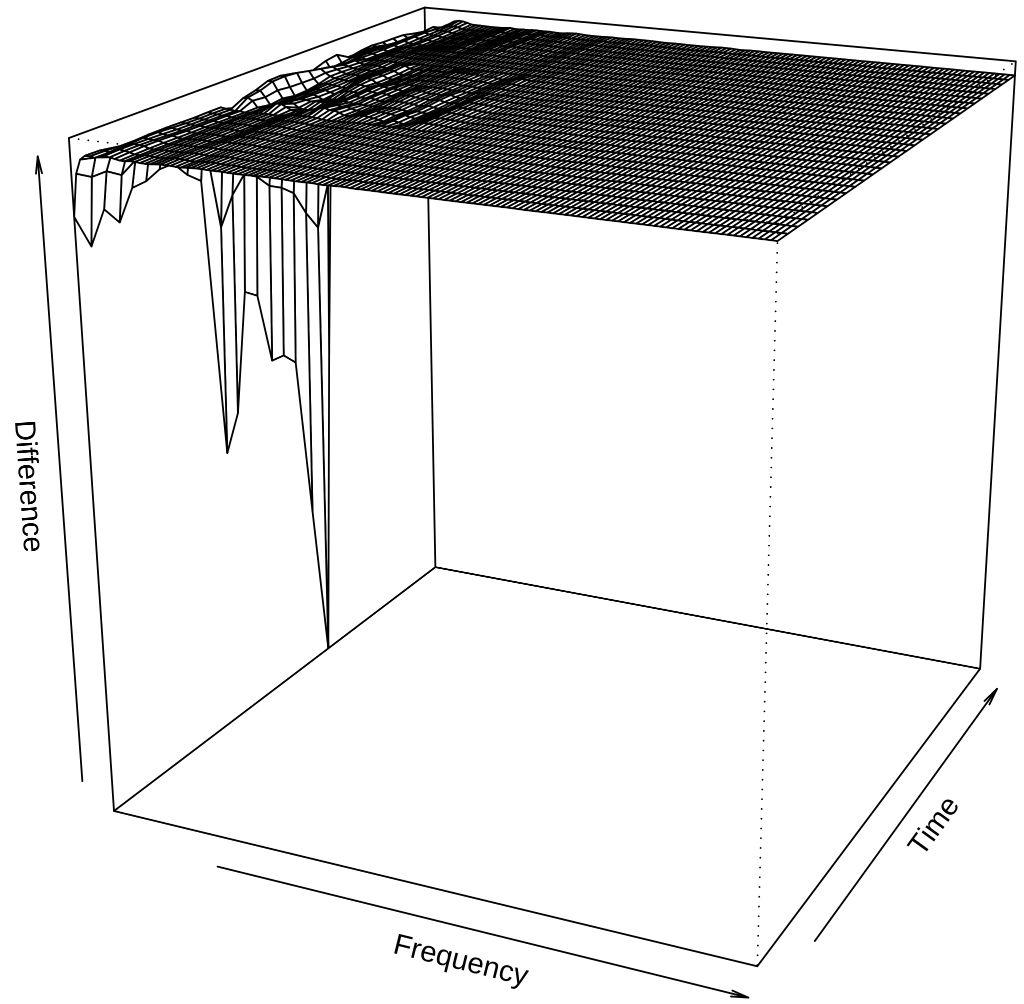
Maximum: 37.979 *10^3 [ms^2]

Robust Short−term Analysis of HRV with AO



Maximum: 224.82 *10^3 [ms^2]

Classical Short−term Analysis without Outliers vs. Robust Short−term Analysis with AO



Minimum: −212.139 *10^3 [ms^2], Maximum: 4.6 *10^3 [ms^2]

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